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On algebras of which every indecomposable representation has an irreducible one as the top or the bottom Loewy constituent

By

HIROYUKI TACHIKAWA

INTRODUCTION. Until the present time the problem how to obtain all indecomposable representations for a given algebra seems to be difficult except the following special types of algebras: semi-simple algebras, group algebras in which the groups have cyclic p-Sylow subgroups (cf. [2]), and generalized uni-serial algebras. In view of this situation the author [11] has recently investigated the class of algebras of strong right cyclic representation type (SRCRT-algebras), in which the class of generalized uniserial algebras is properly contained. But in this generalization the symmetry between left ideal structure and right ideal structure is missing. On the other hand MORITA [7] has established a characterization of QF-1 algebras, and he has shown by an example [7, 2.1] that there is no intimate connection between QF-1 algebras and QF-3 algebras. His example is not a SRCRT-algebra; hence, taking up an interesting property which is satisfied by the example, he has propounded to the author the following question: What structure has an algebra A in which for any indecomposable representation (in the ground field P) either the top constituent in the upper Loewy series or the bottom constituent in the lower Loewy series is irreducible? It may happen that for such an algebra an indecomposable left module L is not homomorphic to a primitive left ideal (an indecomposable direct summand of A), but when this is the case, the dual (representation) module $L^* = \mathrm{Hom}(L, P)$ is homomorphic to a primitive right ideal: that is to say, every indecomposable module is either epimorphic to an indecomposable projective module or monomorphic to an indecomposable injective module. So such an algebra will be called to be of cyclic-cocyclic representation type (abbreviated by cyclic-cocyclic type). The purpose of this paper is to determine the structure of algebras of cyclic-cocyclic type.

Let A be a finite dimensional associative algebra with a unit element 1, N the radical of A and $e_\varkappa$ (or e_λ, e_μ) the primitive idempotents of A. Then in order that A be of cyclic-cocyclic type it is necessary and sufficient that the following conditions are satisfied:

I. Each of $Ne_\varkappa$ and $e_\lambda N$ is either uni-serial (i.e. its composition series is unique) or a sum (not necessarily direct sum) of two uni-serial ideals.

II. In case $Ne_\varkappa$ is a sum of two uni-serial ideals, if $e_\mu Ne_\varkappa \not\equiv 0 \bmod J$ for a simple subideal J of $Ne_\varkappa$, then $e_\mu A$ is uni-serial. And the same holds for a right ideal $e_\lambda N$ which is a sum of two uni-serial ideals.

In this characterization it seems of interest that there exists an intimate relation between left ideal structure and right ideal structure (cf. the condition II) and the notion "uni-serial" which is introduced by KOETHE [4] still plays an important role.

In §1 we shall consider the case where $N^2=0$. All necessary conditions for algebras of cyclic-cocyclic type are obtained in §2. Using the condition II stated above, §3 is devoted to a discussion of the problem whether any isomorphism between simple subideals of $Ae_\varkappa$ can be extendable or not. The sufficiency proof for the conditions is completed in §4.

Throughout this paper a unit element of A is assumed to act as the identity operator over all A-modules considered, N denotes the radical of A. The denominations and the notations used in [11] will retain the same significances and also knowledge of our paper [11] will be assumed.

§1. The case where $N^2=0$

Let $1=\sum_{\varkappa=1}^{n}\sum_{i=1}^{f(\varkappa)} e_{\varkappa,i}$ be a decomposition of a unit element of A into a sum of mutually orthogonal primitive idempotents, where $Ae_{\varkappa,i} \approx Ae_{\lambda,j}$, if and only if $\varkappa=\lambda$. Put $e=\sum_{\varkappa=1}^{n} e_{\varkappa,1}$; then eAe is the basic algebra of A, and it is known (cf. [6]) that there exist covariant functors between the category of A-modules and the category of eAe-modules, which are inverses of each other. Hence in order that A be of cyclic-cocyclic type it is necessary and sufficient that eAe be of the same type. By this reason hereafter we shall always assume that A is a self-basic algebra (i.e. $f(\varkappa)=1$ for all $\varkappa$).

Now we shall consider the case where $N^2=0$ (for the meaning of N cf. Introduction).

LEMMA 1.1. *Let A be an algebra of cyclic-cocyclic type with $N^2=0$. If $e_\varkappa N$ is not simple, then $e_\varkappa N=e_\varkappa u e_\lambda A \oplus e_\varkappa v e_\mu A$, and Ae_λ and Ae_μ are both uni-serial, where e_λ and e_μ are not necessarily different from each other.*

PROOF. Since $N^2=0$, $e_\varkappa N$ is completely reducible and we have a direct decomposition of $e_\varkappa N$: $e_\varkappa N=e_\varkappa u_1 e_\lambda A \oplus e_\varkappa u_2 e_\mu A \oplus \cdots$. Let M be an interlacing of Ae_λ and Ae_μ obtained by using a map ϑ: $xe_\varkappa u_1 e_\lambda \to xe_\varkappa u_2 e_\mu$ for all $x \in A$ as a lacing isomorphism. We shall show that M is directly indecomposable. Suppose that M is decomposed into $Ae_\lambda t_1$ and $Ae_\mu t_2$. Then we have $e_\lambda \alpha = e_\lambda x e_\lambda t_1 + e_\lambda y e_\mu t_2$ and $e_\mu \beta = e_\mu x' e_\lambda t_1 + e_\mu y' e_\mu t_2$, where α and β are injections of Ae_λ and Ae_μ into M respectively. Since $[M]=[Ae_\lambda]+[Ae_\mu]-1$, either a natural homomorphism: $Ae_\lambda \to Ae_\lambda t_1$ or a natural homomorphism: $Ae_\mu \to Ae_\mu t_2$ is an isomorphism. Suppose $Ae_\lambda \approx Ae_\lambda t_1$. Then, as $Ae_\mu t_2$ is not isomorphic to Ae_μ, $e_\lambda x e_\lambda$ must be a regular element of $e_\lambda Ae_\lambda$. From $u_1 e_\lambda \alpha = u_2 e_\mu \beta$ we obtain $u_1 = u_2 e_\mu x' e_\lambda (e_\lambda x e_\lambda)^{-1}$, for $u_1 e_\lambda x e_\lambda = u_2 e_\mu x' e_\lambda$. But

this contradicts the direct decomposition of $e_\varkappa N$. Thus M is indecomposable. Now, if Ne_λ or Ne_μ is not simple, both the top Loewy constituent of M and the bottom Loewy constituent[1]) of M are not simple and this contradicts the assumption that A is of cyclic-cocyclic type. Hence Ae_λ, Ae_μ, ... are uni-serial. Then it follows from Propositions 3.1, 3.3 and 3.5 of [*11*] that the number of $u_1, u_2, \ldots$ is at most two, because otherwise A would not be of cyclic-cocyclic type. This completes the proof.

REMARK. In Lemma 1.1 the case where $e_\lambda = e_\mu$ does not happen, if P is an algebraically closed field. In fact, if $e_\lambda = e_\mu$, it follows by theorem 2.1 of [*3*] that A is of unbounded type, and A is, of course, not of cyclic-cocyclic type. (For the case where P is not algebraically closed, see [*12*].)

Then by the duality of modules we obtain

COROLLARY 1.2. *In Lemma* 1.1 *right* A*-modules* $e_\varkappa A/e_\varkappa u e_\lambda A$ *and* $e_\varkappa A/e_\varkappa v e_\mu A$ *are injective.*

COROLLARY 1.3. *Let* A *be an algebra of cyclic-cocyclic type with* $N^2=0$. *Then every indecomposable left (resp. right)* A*-module whose top constituent is not simple is obtained by interlacing of two uni-serial primitive ideals, and is injective.*

COROLLARY 1.4. *In Lemma* 1.1 *every isomorphism between simple submodules of* $e_\varkappa A$ *can be extendable to an automorphism which is induced by multiplication with a regular element of* $e_\varkappa A e_\varkappa$ *from the left hand.*

PROOF. Assume that ϑ is an isomorphism of $e_\varkappa u e_\lambda A$ onto $e_\varkappa u' e_\lambda A$ and $\vartheta(u) = u'$. Then by Lemma 1.1 Ae_λ is uni-serial, i.e. Ne_λ is simple, and $Ne_\lambda = Ae_\varkappa u e_\lambda$. Hence there exists a regular element $e_\varkappa r e_\varkappa$ such that $e_\varkappa r e_\varkappa u = u'$. Thus the conclusion is obvious.

PROPOSITION 1.5. *Let* A *be an algebra with* $N^2=0$. A *is of cyclic-cocyclic type if the following two conditions are satisfied*:

I. $Ne_\varkappa$ *(resp.* $e_\lambda N$*) is isomorphic either to a simple module or to a direct sum of two simple modules.*

II. If $Ne_\varkappa$ *(resp.* $e_\lambda N$*) is not simple and* $e_\mu N e_\varkappa$ *(resp.* $e_\lambda N e_\eta$*)* $\neq 0$, *then* $e_\mu A$ *(resp.* Ae_η*) is uni-serial.*

PROOF. Let M be a left A-module. We shall prove that every indecomposable direct summand of M has a simple module as the top Loewy constituent or the bottom Loewy constituent. Let $e_{\varkappa_i} g_i$, $i = 1, 2, \ldots, n$, be a system of generators of M such that any $Ae_{\varkappa_i} g_i$ is not contained in $\sum_{j \neq i} Ae_{\varkappa_j} g_j$. Assuming that for $n \leqq r$ every indecomposable direct summand of M has a simple module as the top constituent or the bottom constituent, we shall proceed to the proof by induction on n. If an $Ae_{\varkappa_1} g_1$ is simple or injective, then it must be a direct summand of M, and hence our conclusion follows readily from the assumption of induction. Therefore we may assume that all $Ae_{\varkappa_i} g_i$ are neither simple nor injective. Let us denote by U a sum of $Ae_{\varkappa_i} g_i$, $i \geqq 2$, and by D_j, $j = 2, 3, \ldots$, indecomposable direct summands of U,

[1]) The top (resp. bottom) Loewy constituent of a left A-module M means the top (resp. bottom) constituent of upper (resp. lower) Loewy Series.

i.e. $U=D_2\oplus D_3\oplus\cdots$. Then according to Corollary 1.3 we may assume that all D_j are quasi-primitive (for the definition cf. [11]), and hence $Ae_{\varkappa_1}g_1$ and D_j, $j=2, 3, \ldots$ are isomorphic to primitive ideals which are not injective. Now we shall distinguish two cases.

1. Suppose that $Ae_{\varkappa_1}g_1$ is isomorphic to a uni-serial primitive ideal which is not injective. Then every D_j is also isomorphic to uni-serial primitive ideal, for if certain D_j is not uni-serial, it follows from Corollary 1.2 that any simple submodule of D_j is not isomorphic to $Ne_{\varkappa_1}g_1$ (cf. Corollary 2.2 of [5]) and D_j must be a direct summand of M. Thus we may assume that M is a sum of uni-serial modules. On the other hand by Lemma 1.1 it is easily seen that the class of all uni-serial A-modules satisfies conditions III and IV of Theorem 5.4 of [11]. Hence M is decomposed into a sum of modules whose bottom constituents are simple.

2. Suppose that $Ae_{\varkappa_1}g_1$ is isomorphic to $Ae_{\varkappa_1}$ which is not uni-serial; then according to Corollary 1.2 all D_j are isomorphic to $Ae_{\varkappa_1}$, for otherwise at least one D_j would be a direct summand of M. Let σ_j be projections: $Ae_{\varkappa_1}g_1\cap U\to D_j$ and S a simple submodule of $Ae_{\varkappa_1}g_1\cap U$. Let σ_j be the restrictions of σ_j to S. Then by Corollary 1.4 every σ_j is extendable to a homomorphism τ_j: $Ae_{\varkappa_1}g_1\to D_j$. Let $Ae_{\varkappa_1}g_j'=\left\{x-\sum_{j\geqq 2}\tau_j(x)\,|\,x\in Ae_{\varkappa_1}g_1\right\}$. Then M is a sum of $Ae_{\varkappa_1}g_1'$ and U, and $Ae_{\varkappa_1}g_1'$ is injective or simple, for $Ae_{\varkappa_1}g_1$ is isomorphic to a proper homomorphic image of $Ae_{\varkappa_1}$. Since $Ae_{\varkappa_1}g_1'$ is not contained in U, $Ae_{\varkappa_1}g_1'$ must be a direct summand of M and the conclusion of this lemma is obtained from the assumption of induction.

§ 2. Necessary conditions

The following lemmas are essentially proved in Proposition 2.3 of [11].

LEMMA 2.1. *Let M_1 and M_2 be uni-serial left A-modules such that the composition length of M_2 equals to that of M_1, and M_1 and M_2 are homomorphic to uni-serial primitive ideals $Ae_\varkappa$ and Ae_λ respectively (here A is not assumed to be of cyclic-cocyclic type). Let ϑ be an isomorphism: $S^1(M_2)\to S^1(M_1)$. Then the interlacing module of M_1 and M_2 obtained by using ϑ as the lacing isomorphism is directly decomposable if and only if ϑ is extendable to a isomorphism $M_2\to M_1$.*

LEMMA 2.2. *In Lemma 2.1, if we assume that $e_\varkappa\neq e_\lambda$, in place of assuming that $Ae_\varkappa$ and Ae_λ is uni-serial, then the interlacing module is indecomposable.*

PROOF. Let L be an interlacing module obtained by using ϑ as the lacing isomorphism. Suppose that L is decomposed into L_1 and L_2. Then from the proof of Proposition 2.3 of [11] it will be seen that we have only to show that L_1 and L_2 are uni-serial. Denote by p_1 and p_2 the composition length of L_1 and L_2 respectively. If $p_1>p_2$, then $p_2=p-1$, where $p=[M_1]$ and we may assume that $M_1/N^{p-1}M_1\approx L_1/N^{p-1}L_1$ and $M_2/N^{p-1}M_2\approx L_2/N^{p-1}L_2$ for it holds that $L/N^{p-1}L\approx M_1/N^{p-1}M_1\oplus M_2/N^{p-1}M_2$.

Then by calculation of the composition length of L we obtain that $N^{p-1}L_2=0$, $N^{p-1}L_1$ is simple, and hence L_1 is uni-serial.

PROPOSITION 2.3. *Let A be an algebra of cyclic-cocyclic type with $N^3=0$. If $Ne_\varkappa = Ae_\lambda u + Ae_\eta v$, then $Ae_\lambda u$ and $Ae_\eta v$ are uni-serial.*

PROOF. Suppose that $Ne_\lambda u \approx \bar{A}\bar{e}_\mu \oplus \bar{A}\bar{e}_{\mu'}$, consequently $Ne_\lambda/N^2e_\lambda \approx \bar{A}\bar{e}_\mu \oplus \bar{A}\bar{e}_\mu$, where $\bar{A}=A/N$. Then Lemma 1.1 implies that $e_\mu A$ and $e_{\mu'}A$ are both uni-serial, and hence from the assumption it follows that $S^2(e_\mu A) \approx S^2(e_{\mu'}A) \approx e_\lambda A/e_\lambda N^2$, where $S^2(e_\mu A)$ and $S^2(e_{\mu'}A)$ denote the annihilators of N^2 in $e_\mu A$ and $e_{\mu'}A$ respectively. Let Θ be an isomorphism: $S^2(e_\mu A) \to S^2(e_{\mu'}A)$ and ϑ the restriction of Θ to $S^1(e_\mu A)$. Then if we choose Θ suitably, the interlacing of $e_\mu A$ and $e_{\mu'}A$ obtained by using ϑ as lacing isomorphism is indecomposable. To prove this we shall distinguish two cases.

1. $e_\mu = e_{\mu'}$. Then it holds that

$$(e_\mu Ne_\lambda/e_\mu N^2 e_\lambda : \bar{e}_\lambda \bar{A}\bar{e}_\lambda) = 1 \quad \text{and} \quad (e_\mu Ne_\lambda/e_\mu N^2 e_\lambda : \bar{e}_\mu \bar{A}\bar{e}_\mu) = 2.$$

Hence we can find an automorphism Θ: $S^2(e_\mu A) \to S^2(e_\mu A)$ such that it has no extension. Let ϑ be the restriction of Θ to $S^1(e_\mu A)$. Since $e_\mu A$ is uni-serial, it follows Lemma 2.1 that $\mathrm{Int}_\vartheta(e_\mu A, e_\mu A)$ is indecomposable.

2. $e_\mu \neq e_{\mu'}$. Then any isomorphism Θ: $S^2(e_\mu A) \to S^2(e_{\mu'}A)$ is not extendable. Hence from Lemma 2.2 $\mathrm{Int}_\vartheta(e_\mu A, e_{\mu'}A)$ is indecomposable, if ϑ is the restriction of Θ to $S^1(e_\mu A)$.

Now, in both cases Θ is an extension of ϑ and hence it follows from Lemma 1.3 of [*11*] that the bottom constituent of $\mathrm{Int}\,\vartheta(e_\mu A, e_{\mu'}A)$ is not simple. Thus $\mathrm{Int}_\vartheta(e_\mu A, e_{\mu'}A)$ has non-simple modules as the top constituent and the bottom constituent. This contradicts the hypothesis that A is of cyclic-cocyclic type.

PROPOSITION 2.4. *Let A be an algebra of cyclic-cocyclic type with $N^3=0$. If L is a left A-module such that L/N^2L is uni-serial, then L is itself uni-serial.*

PROOF. Let L be homomorphic to $Ae_\varkappa$. Suppose that L is not uni-serial. Then from Proposition 2.3 we may assume that $Ne_\varkappa/N^2e_\varkappa \approx \bar{A}\bar{e}_\lambda$ and $N^2e_\varkappa \approx \bar{A}\bar{e}_\mu \oplus \bar{A}\bar{e}_\nu$. Put $N^2e_\varkappa = Ae_\mu u \oplus Ae_\nu v$; then the dual representation modules M_1 and M_2 of $Ae_\varkappa/Ae_\mu u$ and $Ae_\varkappa/Ae_\nu v$ are uni-serial, and we have $M_1/M_1N \approx \bar{e}_\nu \bar{A}$, $M_2/M_2N \approx \bar{e}_\mu \bar{A}$ and $S^2(M_1) = M_1N \approx S^2(M_2) = M_2N \approx (Ae_\varkappa/N^2e_\varkappa)^*$, where $(Ae_\varkappa/N^2e_\varkappa)^*$ denotes the dual representation module of $Ae_\varkappa/N^2e_\varkappa$. Let us denote by Θ an isomorphism of $S^2(M_1)$ onto $S^2(M_2)$ and by ϑ the restriction of Θ to $S^1(M_1)$.

If $e_\mu \neq e_\nu$, it is clear that Θ has no extension. Hence by Lemma 2.2 $\mathrm{Int}_\vartheta(M_1, M_2)$ is indecomposable and further by Lemma 1.3 of [*11*] the bottom constituent of it is not simple. Since the top constituent of $\mathrm{Int}_\vartheta(M_1, M_2)$ is clearly non-simple, this leads us to a contradiction.

Next suppose that $e_\mu = e_\nu$. Then, since $Ne_\varkappa$ is homomorphic to Ae_λ and $(e_\mu Ne_\lambda/e_\mu N^2e_\lambda : \bar{e}_\mu \bar{A}\bar{e}_\mu) = 2$, we have from Lemma 1.1 that $(e_\mu Ne_\lambda/e_\mu N^2e_\lambda : \bar{e}_\lambda \bar{A}\bar{e}_\lambda) = 1$. Further it holds that

$$(e_\mu N^2 e_\varkappa : \bar{e}_\varkappa \bar{A}\bar{e}_\varkappa) = 1,$$

for if $(e_\mu N^2 e_\varkappa : \bar{e}_\varkappa \bar{A}\bar{e}_\varkappa) \geqq 2$, then together with the condition $(e_\mu N^2 e_\varkappa : \bar{e}_\mu \bar{A}\bar{e}_\mu) \geqq 2$

Theorem 2 of [12] implies that A is an algebra of unbounded representation type. Hence by Proposition 2.3 it holds that

$$(\bar{e}_\varkappa \bar{A} \bar{e}_\varkappa \colon P) = (\bar{e}_\lambda \bar{A} \bar{e}_\lambda \colon P) = 2(\bar{e}_\mu \bar{A} \bar{e}_\mu \colon P),$$

where P is the ground field of A. Now if $e_\mu A$ is not uni-serial, from the above equation it follows that $e_\mu N/e_\mu N^2 \approx \bar{e}_\lambda \bar{A}$, $e_\mu N^2 \approx \bar{e}_\varkappa \bar{A} \oplus \bar{e}_{\varkappa'} \bar{A}$, where $e_\varkappa \neq e_{\varkappa'}$. But as is shown in the first case, this leads us to a contradiction. Hence $e_\mu A$ is uni-serial, i.e. $e_\mu N/e_\mu N^2 \approx \bar{e}_\lambda \bar{A}$ and $e_\mu N^2 \approx \bar{e}_\varkappa \bar{A}$. Then again from the above equation, we can choose an automorphism T of $S^2(e_\mu A) = e_\mu N$ so that it is maximal, and by using τ, which is the restriction of T to $S^1(e_\mu A) = e_\mu N^2$, as the lacing isomorphism we obtain an indecomposable interlacing module of two copies of $e_\mu A$ such that its top constituent and bottom constituent are not simple. Hence we conclude that L is uni-serial.

In Proposition 2.4 the condition that $N^3 = 0$ can be released.

PROPOSITION 2.5. *Let A be of cyclic-cocyclic type. If L is a left A-module such that L/N^2L is uni-serial, then L is uni-serial.*

PROOF. Assume that $N^p L \neq 0$ and $L/N^p L$ is uni-serial; then $N^{p-2}L/N^p L$ is uni-serial. Hence it follows from Proposition 2.4 that $N^{p-2}L/N^{p-1}L$ is uni-serial and equivalently $N^p L/N^{p-1}L$ is simple. Thus by means of induction our conclusion is proved.

Then applying Proposition 2.5 to Proposition 2.3 the condition that $N^3 = 0$ can be also released from Proposition 2.3 and we have

PROPOSITION 2.6. *Let A be of cyclic-cocyclic type. If $Ne_\varkappa/N^2 e_\varkappa$ is not simple, then $Ne_\varkappa$ is a sum of two uni-serial ideals.*

PROPOSITION 2.7. *In Proposition 2.6 let $Ne_\varkappa$ be a sum of two uni-serial ideals $Ae_\lambda u$ and $Ae_\mu v$. Then we can take $e_\lambda u$ and $e_\mu v$ such that the intersection of $Ae_\lambda u$ and $Ae_\mu v$ is either zero or simple (i.e. $Ae_\varkappa$ has a unique minimal subideal).*

PROOF. Assume that $Ne_\varkappa$ is indecomposable, then $S^1(Ne_\varkappa)$ is simple and by Proposition 2.4 $S^2(Ne_\varkappa)$ is not uni-serial. Let $(Ne_\varkappa)^* N = u' e_{\lambda'} A + v' e_{\mu'} A$, and put Annih $(u' e_{\lambda'} A : Ne_\varkappa) = Ae_\lambda u$ and Annih $(v' e_{\mu'} A : Ne_\varkappa) = Ae_\mu v$, then $Ae_\lambda u \cap Ae_\mu v$ is simple.

PROPOSITION 2.8. *In Proposition 2.7 assume that $e_\eta N^{p-1} e_\lambda u \not\equiv 0 \bmod. N^p e_\lambda u + Ne_\mu v$, $N^p e_\lambda u + Ne_\mu v \neq 0$ for $p \geqq 2$. Let us denote by s $(= e_\varkappa s)$ a generator of $Ae_\varkappa/N^p e_\lambda u + Ne_\mu v$ and by $Ae_\tau t$ a left A-module such that $Ne_\tau t$ is isomorphic to $\bar{A}\bar{e}_\eta$. Then any isomorphism $\vartheta \colon Ne_\tau t \to N^{p-1} e_\varkappa s$ is extendable to a monomorphism: $Ae_\tau t \to Ae_\varkappa s$.*

PROOF. We shall first consider the case where $e_\tau = e_\varkappa$. Since $Ae_\tau t$ is considered as a residue class module of $Ae_\varkappa$, by Corollary 1.2 $Ae_\tau t$ as well as $N^{p-2} e_\lambda u e_\varkappa s$ is injective relative to N^2. Hence ϑ is extendable.

Hence we may assume $e_\tau \neq e_\varkappa$. Suppose that ϑ is maximal and let M be an interlacing of $Ae_\tau t$ and $Ae_\varkappa s$ obtained by using ϑ as the lacing isomorphism; then M is directly indecomposable. To prove this, suppose that M is decomposed into M_1 and M_2, where M_1 and M_2 is homomorphic to $Ae_\varkappa$ and Ae_τ respectively. A generator w of M_1 is expressed as follows:

$e_\varkappa w = e_\varkappa x_1 e_\tau t + e_\varkappa x_2 e_\varkappa s$, x_1 and $x_2 \in A$. Then $e_\varkappa x_2 e_\varkappa$ is a regular element of $e_\varkappa A e_\varkappa$, because $e_\tau \neq e_\varkappa$. Therefore

$$N w = N M_1 = N e_\varkappa x_2 e_\varkappa s.$$

Since $e_\varkappa x_2 e_\varkappa$ is a regular element, the right term of the above equations is the unique maximal submodule of $A e_\varkappa s$, and at the same time $N w$ is a unique maximal submodule of M_1. Hence $S^1(M_1) = S^1(A e_\varkappa s)$. On the other hand, since ϑ is maximal, $S^1(A e_\varkappa s) = S^1(M)$. But this is a contradiction. Therefore $\mathrm{Int}\,\vartheta(A e_\varkappa s, A e_\tau t)$ is indecomposable. As the top constituent of M and the bottom constituent of it are not simple, this contradicts to the fact that A is of cyclic-cocyclic type. Thus ϑ must be extendable.

PROPOSITION 2.9. *In Proposition* 2.8 $e_\eta A$ *is a uni-serial A-module.*

PROOF. In the proof of Proposition 2.7 the image of an extension of ϑ is contained in the S^2-part of $A e_\varkappa / N^p e_\lambda u + N e_\mu v$. Let $N^{p-2} e_\lambda u / N^{p-1} e_\lambda u$ be isomorphic to $\bar{A} \bar{e}_\eta$. Since the S^2-part of $A e_\varkappa / N^p e_\lambda u + N e_\mu v$ is isomorphic to $N^{p-2} e_\lambda u / N^p e_\lambda u$, by Proposition 2.8 we have $e_\xi = e_\tau$. Hence every simple constituents of $e_\eta N / e_\eta N^2$ is isomorphic to $\bar{e}_\tau \bar{A}$ and the multiplicity of $\bar{e}_\zeta \bar{A}$ in $e_\eta N / e_\eta N^2$ is one, for if the multiplicity equals to two, we can choose ϑ so as to be a maximal homomorphism and this opposes to the result of Proposition 2.8. Then it follows from Proposition 2.5 that $e_\eta A$ is uni-serial.

Then combining Lemma 1.1, Proposition 2.6 and Proposition 2.8 we can easily prove

PROPOSITION 2.10. *Let A be of cyclic-cocyclic type. Let* $N e_\varkappa$ *be a sum of two uni-serial ideals and J a simple subideal of* $N e_\varkappa$. *If* $e_\eta N e_\varkappa \not\equiv 0 \bmod J$, *then* $e_\eta A$ *is uni-serial.*

In concluding this section we sum up the results of this section.

PROPOSITION 2.11. *Let A be an algebra of cyclic-cocyclic type. Then the following conditions are satisfied*:

I. $N e_\varkappa$ *(resp.* $e_\lambda N$*) is either uni-serial or a sum of two uni-serial ideals.*

II. In case $N e_\varkappa$ *is a sum of two uni-serial ideals, if* $e_\eta N e_\varkappa \not\equiv 0 \bmod J$ *for a simple subideal J of* $N e_\varkappa$, *then* $e_\eta A$ *is uni-serial. And the same holds for* $e_\lambda N$ *which is a sum of two uni-serial ideals.*

§ 3. Preparations for the sufficiency proof

Let A be an algebra over a field P and with a unit element 1. We shall assume for all propositions in this section that A satisfies the conditions stated in Proposition 2.11.

PROPOSITION 3.1. *Let* $A e_\varkappa$ *be a primitive left ideal of A. If* $N e_\varkappa$ *is a sum of two uni-serial ideals, and if* $A e_\varkappa$ *has a unique minimal subideal, then* $A e_\varkappa$ *is injective.*

PROOF. Suppose $A e_\varkappa$ is not injective; then for a primitive right ideal $e_\lambda A$, $A e_\varkappa$ is isomorphic to a proper submodule of $(e_\lambda A)^*$, where $(e_\lambda A)^*$ denotes the dual (representation) module of $e_\lambda A$. Since by the condition II of Proposition 2.11 every proper submodule of $(e_\lambda A)^*$ with a simple top constituent

is uni-serial, $Ae_\varkappa$ must be uni-serial. But this contradicts to the assumption. Thus $Ae_\varkappa \approx (e_\lambda A)^*$ and $Ae_\varkappa$ is injective (cf. [8]).

PROPOSITION 3.2. *Let M be a homomorphic image of $Ae_\varkappa$ and S a simple submodule of M. If the S^1-part of M is non-simple, then we can take elements u and v of $Ne_\varkappa$ such that*

$$\operatorname{Ker}\vartheta = N^p u + N^t v, \quad \text{and} \quad \vartheta(N^{p-1}u + N^t v) = S,$$

where ϑ is the natural homomorphism of $Ae_\varkappa$ onto M.

PROOF. Let us denote by $e_\varkappa b$ a generator of M. $Ne_\varkappa b$ is an interlacing module of two uni-serial module and each of them is injective relative to a power of N (its exponent is equal to the composition length of it (cf. [11], p. 212 and the condition II of Proposition 2.11). Thus $Ne_\varkappa b$ is a direct sum of uni-serial modules $Ae_\lambda u'b$ and $Ae_\mu v'b$. Then it holds that $Ne_\varkappa = Ae_\lambda u' + Ae_\mu v'$, for $\operatorname{Ker}\vartheta \subseteq N^2 e_\varkappa$ and $Au' + Av' + \operatorname{Ker}\vartheta = Ne_\varkappa$ and hence u' and v' are not contained in $N^2 e_\varkappa$ (cf. [14], Lemma 5). If $N^p e_\lambda u'b = 0$, $N^{p-1}e_\lambda u'b \neq 0$, $N^t e_\mu v'b = 0$, $N^{t-1}e_\mu v'b \neq 0$, then $\operatorname{Ker}(Ae_\varkappa \to Ae_\varkappa b) = N^p e_\lambda u' + N^t e_\mu v'$. This is the first conclusion.

Since $Ae_\lambda u'b$ and $Ae_\mu v'b$ are injective relative to N^p and N^t. Hence we can alter generators u' and v' of $Ne_\varkappa b$ to u and v such that $N^{p-1}ub$ or $N^{p-1}vb$ is identical to S and $Ne_\varkappa b = Aub \oplus Avb$. This completes the proof.

Let M and M' be homomorphic to $Ae_\varkappa$ and Ae_λ respectively, and suppose that they have non-simple S^1-part. Let S and S' be simple submodules of M and M' respectively. Then according to Proposition 3.2 we can select elements $ue_\varkappa$, $ve_\varkappa$, ue_λ, ve_λ and obtain the following exact sequences:

$$0 \to N^m u + N^n v \to Ae_\varkappa \xrightarrow{\eta} M \to 0,$$
$$0 \to N^m u + N^n v \to N^{m-1}u + N^n v \xrightarrow{\eta} S \to 0,$$
$$0 \to N^{m'} u' + N^{n'} v' \to Ae_\lambda \xrightarrow{\eta} M \to 0,$$
$$0 \to N^{m'} u' + N^{n'} v' \to N^{m'-1}u' + N^{n'} v' \xrightarrow{\eta'} S' \to 0.$$

Assume that S and S' are isomorphic to each other. Then we have

PROPOSITION 3.3. *If $m < m'$, then every isomorphism of S onto S' can be extendable to a homomorphism of M into M'.*

PROOF. By $e_\varkappa a$ and $e_\lambda b$ let us denote images of $e_\varkappa$ and e_λ by natural homomorphisms η, η'. Since $Ne_\varkappa a = Aua \oplus Ava$ and $Ne_\lambda b = Au'b \oplus Av'b$, ϑ: $N^{m-1}ua\,(=S) \to N^{m'-1}u'b\,(=S')$ is considered as an isomorphism $N^{m-1}ua + Ava/Ava \to N^{m'-1}u'b$. Then ϑ is extendable to a monomorphism Θ: $Ae_\varkappa a/Ava \to Nu'b$, for $Nu'b$ is injective relative to $N^{m'-1}$ and $m < m'$. Hence by an obvious way we obtain a homomorphism of $Ae_\varkappa a$ into $Ae_\lambda b$ which is an extension of ϑ.

PROPOSITION 3.4. *If $m = m'$ and if $n \geqq n'$, then every isomorphism of S onto S' can be extendable to a homomorphism of M into M'.*

PROOF. We shall divide the proof into two parts.

(i) Assume $m \geqq n$. From the assumption $m = m'$ we obtain $e_\varkappa = e_\lambda$. Since $Nu \cap Av \subseteq N^m u + N^n v$ and $Nu' \cap Av' \subseteq N^{m'}u' + N^{n'}v'$, ϑ may be considered as

a map: $N^{m-1}u+Av/N^mu+Av\to N^{m'-1}u'+Av'/N^{m'}u'+Av'$. Then, since $Ae_\varkappa/N^mu\ Av$ and $Ae_\lambda/N^mu'+Av'$ are injective relative to N^{m+1}, ϑ is extendable to an isomorphism ϑ': $Ae_\varkappa/N^mu+Au\to Ae_\varkappa/N^mu'+Av'$. Put

$$\vartheta'(e_\varkappa \bmod N^mu+Av)=e_\varkappa re_\varkappa \bmod N^mu'+Av',$$

then it holds that

$$ur\equiv au' \bmod N^{m'}u'+Av' \quad \text{and} \quad vr\equiv 0 \bmod N^{m'}u'+Av',$$

and au' is a generator of Au'. Now we shall consider a map Θ: $Ae_\varkappa/N^mu+N^nv\to Ae_\varkappa/N^{m'}u'+N^{n'}v'$, which is defined by

$$\Theta(xe_\varkappa \bmod N^mu+N^nv)=xe_\varkappa re_\varkappa \bmod N^{m'}u'+N^{n'}v'$$

for all $x\in A$. Θ is clearly an extension of ϑ, and we must verify that Θ is well defined. It follows however from the above equations that

$$N^mur\equiv N^mau' \bmod N^{2m}u'+N^mv'$$

and from $n'<n\leqq m$ we have $N^mur\leqq N^{m'}u'+N^{n'}v'$. On the other hand it holds $N^nvr\equiv 0 \bmod N^{m'-n}u'+N^nv'$, hence $N^nvr\leqq N^{m'}u'+N^{n'}v'$. Thus we obtain $(N^mu+N^nv)r\leqq N^{m'}u+N^{n'}v$. This completes the proof for this case.

(ii) Assume $m<n$. By the same reason as in the first case ϑ may be considered as a map: $N^{m-1}u+N^nv/N^mu+N^nv\to N^{m'-1}u'+N^{n'}v'/N^{m'}u'+N^{n'}v'$. Since $Au+N^nv/N^mu+N^nv$ and $Au'+N^{n'}v'/N^{m'}u'+N^{n'}v'$ are injective relative to N^m, ϑ is extendable to an isomorphism ϑ':

$$Au+N^nv/N^mu+N^nv\to Au'+N^{n'}v'/N^{m'}u'+N^{n'}v'.$$

Put $\vartheta'(u \bmod N^mu+N^nv)=au' \bmod N^{m'}u'+N^{n'}v'$, then a is a regular element of $e_\mu Ae_\mu$ when Au is a homomorphic image of Ae_μ. From the condition II of Proposition 2.11 it follows that $e_\mu A$ and consequently $e_\mu N$ are uni-serial right ideal, and au' and u are both generator of $e_\mu N$. Hence there exists a regular element r of $e_\varkappa Ae_\varkappa$ such that

(1) $$au'=ur.$$

Then by using this r we can define a homomorphism Θ which is an extension of Θ' (consequently ϑ) as follows:

$$\Theta(xe_\varkappa \bmod N^mu+N^nv)=xe_\varkappa re_\varkappa \bmod N^{m'}u'+N^{n'}v',$$

for all $x\in A$. Therefore it will remain to prove that Θ is well defined. By (1) we obtain $N^mur=N^mau'=N^mu'$. On the other hand it follows from $n>m'$, n' and $N^nvr<N^{n+1}e_\varkappa$ that $N^nvr\leqq N^{m'}u'+N^{n'}v'$. Hence

$$(N^mu+N^nv)r=N^{m'}u+N^{n'}v.$$

Thus the proof is completed.

§ 4. Sufficiency proof

In this section we shall give a sufficiency proof of the conditions in our main theorem (cf. Introduction); we shall assume throughout this section that the conditions I and II of Proposition 2.11 are satisfied by an algebra A considered in the following. Then we have first

PROPOSITION 4.1. *The totality of uni-serial left A-modules satisfies the conditions II, III and IV of Theorem* 5.4 *of* [*11*].

PROOF. We have only to ascertain the validity of the condition II of Theorem 5.4 of [*11*]. Suppose it is not satisfied; then we can construct a left A-module M which is an interlacing of two uni-serial modules and has a uni-serial S^2-part. Then M^*/M^*N^2 is uni-serial but M^* is not uni-serial, and this contradicts the condition I of Proposition 2.11 (cf. also Proposition 2.5).

Let M be a left A-module such that the top Loewy constituent is isomorphic to a direct sum of simple left A-modules $\bar{A}\bar{e}_{\varkappa_i}$, $i=1, 2, \ldots, n$, where $e_{\varkappa_i}$ is not necessarily different from the others. Now we shall prove that either the top Loewy constituent of any indecomposable direct summand of M or the bottom Loewy constituent is simple.

If all $Ae_{\varkappa_i}$ are uni-serial ideals, M is considered as an interlacing of uni-serial modules each of which is a homomorphic image of $Ae_{\varkappa_i}$, $i=1, 2, \ldots, s$, and then according to Proposition 4.1 every direct summand of M has a simple module as the bottom Loewy constituent. Hence we may assume that there exists a primitive idempotent, say $e_{\varkappa_1}$ for which $Ae_{\varkappa_1}$ is not uni-serial. Then the following proposition holds.

PROPOSITION 4.2. *Let M be a left A-module and let* $M/NM \approx \sum_{i=1}^{s} \bar{A}\bar{e}_{\varkappa_i}$. *If $s \geqq 2$ and if $Ae_{\varkappa_1}$ is neither uni-serial nor simple, then M is directly decomposable and has a direct summand whose top Loewy constituent is isomorphic to $\bar{A}\bar{e}_{\varkappa_1}$.*

If this proposition is proved, our pursose will be accomplished by Proposition 4.1.

PROOF. This is done by double induction. Let $N^{r+1}M$ be zero-module. We shall assume first that this proposition holds for all left A-module L such that $N^rL=0$. As was shown in § 2, this assumption is really satisfied for $r=2$.

Next we assume that this proposition holds for a left A-module L (and hence every directly indecomposable direct summand of L has a simple module as its top Loewy constituent or its bottom Loewy constituent), if L is annihilated by N^{r+1} and if the top Loewy constituent of L is a sum of at most $s-1$ simple left A-modules.

Denote by $\widetilde{M}$ the factor module of M by N^rM. According to the first assumption we have a system of generators $\{\tilde{e}_{\varkappa_1}\tilde{f}_1, \tilde{e}_{\varkappa_2}\tilde{f}_2, \ldots, \tilde{e}_{\varkappa_s}\tilde{f}_s\}$ of $\widetilde{M}$ and a direct decomposition of $\widetilde{M}$ as follows:

$$\widetilde{M} = A\tilde{e}_{\varkappa_1}\tilde{f}_1 \oplus \cdots .$$

Now choose an element $e_{\varkappa_i}f_i$ from each coset $\tilde{e}_{\varkappa_i}\tilde{f}_i$. Then the set $\{e_{\varkappa_1}f_1, e_{\varkappa_2}f_2, \ldots, e_{\varkappa_s}f_s\}$ can be taken as a system of generators of M (cf. [5], Corollary 2.2)

and $Ae_{\varkappa_1}f_1 \cap \sum_{i\geq 2} Ae_{\varkappa_i}f_i \subset S^1(M)$. For let $x \in Ae_{\varkappa_1}f_1 \cap \sum_{i\geq 2} Ae_{\varkappa_i}f_i$, then $x \in N^r M$ and $N^{r+1}M = 0$. On the other hand from the second assumption we have a direct decomposition of $\sum_{i\geq 2} Ae_{\varkappa_i}f_i$: $\sum_{i\geq 2} Ae_{\varkappa_i}f_i = D_1 \oplus D_2 \oplus \cdots \oplus D_{s'}$, $s' \leqq s$, where D_j has a simple module as its top Loewy constituent or its bottom Loewy constituent. Let us denote by p_j the projection: $Ae_{\varkappa_1}f_1 \cap \sum_{i\geq 2} Ae_{\varkappa_i}f_i \to D_j$ with respect to the above direct sum expression. We may assume $\operatorname{Im} p_j \neq 0$ for all j. For otherwise some D_j is a direct summand of M and our proposition follows from the second assumption. Next we may assume that the bottom Loewy constituent of $Ae_{\varkappa_i}f_1$ is not simple. For otherwise unless $Ae_{\varkappa_1}f_1$ is not uni-serial $Ae_{\varkappa_1}f_1$ is isomorphic to $Ae_{\varkappa_1}$ and then since $Ae_{\varkappa_1}$ is not uni-serial it follows by Proposition 3.1 that $Ae_{\varkappa_1}f_1$ is injective and becomes a direct summand of M. Then D_j is isomorphic to a submodule of uni-serial injective module, if the bottom constituent of D_j is simple. For $\operatorname{Im} p_j \approx \operatorname{Coim} p_j$, and $Ae_{\varkappa_1}$ is not uni-serial. Hence D_j is a homomorphic image of a uni-serial primitive ideal. Together with the second assumption this implies that all D_j are homomorphic images of primitive ideals and $s' = s$. Thus we shall denote D_j, $i = 2, \ldots, s$ again $Ae_{\varkappa_i}f_i$, $i = 1, 2, \ldots, s$ respectively and $Ae_{\varkappa_1}f_1 \cap \sum_{i\geq 2} Ae_{\varkappa_i}f_i$ by S. In this case without loss of generality we may assume $Ae_{\varkappa_i}$ is not uni-serial for $i \leqq k$ and uni-serial for $i > k$. Now we shall divide this proof into two parts.

1. S is simple. Then images of p_i, $i = 2, \ldots, s$ are simple and by Proposition 3.2 we can choose elements u_i and v_i of $Ne_{\varkappa_i}$ for $i \leqq k$ and obtain the following exact sequences:

$$0 \to N^{m_i}u_i + N^{n_i}v_i \to Ae_{\varkappa_i} \to Ae_{\varkappa_i}f_i \to 0,$$
$$0 \to N^{m_i}u_i + N^{n_i}v_i \to N^{m_i-1}u_i + N^{n_i}v_i \to p_i(S) \to 0,$$

where $p_1(S) = S$. For $i > k$ we also obtain the following exact sequences:

$$0 \to N^{l_i}e_{\varkappa_i} \to Ae_{\varkappa_i} \to Ae_{\varkappa_i}f_i \to 0,$$
$$0 \to N^{l_i}e_{\varkappa_i} \to N^{l_i-1}e_{\varkappa_i} \to p_i(S) \to 0$$

we shall choose an integer i_0 among $\{1, 2, \ldots, s\}$, for which (according as $Ae_{\varkappa_{i_0}}$ is not uni-serial or uni-serial) m_{i_0} or l_{i_0} is minimum among $\{l_j, m_i\}$ and if $Ae_{\varkappa_{i_0}}$ is especially not uni-serial, $m_{i_0} < l_j$ for all $j > k$ and $n_{i_0} > n_i$ for all i such that $m_i = m_{i_0}$. In followings such $Ae_{\varkappa_{i_0}}f_{i_0}$ is called a minimum element of $\{Ae_{\varkappa_i}f_i\}$ with respect to S (in fact we can introduce a quasi-ordering into $\{Ae_{\varkappa_i}f_i\}$, for which $Ae_{\varkappa_{i_0}}f_{i_0}$ is minimum). Since S is simple and $\sum_{i\geq 2} Ae_{\varkappa_i}f_i$ is a direct sum, we have that $Ae_{\varkappa_{i_0}}f_{i_0} \cap \sum_{i\neq i_0} Ae_{\varkappa_i}f_i$ is simple and $\sum_{i\neq i_0} Ae_{\varkappa_i}f_i$ is a direct sum of $Ae_{\varkappa_i}f_i$, $i \neq i_0$. Let q_i be a projection: $Ae_{\varkappa_{i_0}}f_{i_0} \cap \sum_{i\neq i_0} Ae_{\varkappa_j}f_j \to Ae_{\varkappa_j}f_j$ with respect to the above direct expression of $\sum_{i\neq i_0} Ae_{\varkappa_i}f_i$.

If $Ae_{\varkappa_{i_0}}$ is not uni-serial, it follows by Propositions 3.3 and 3.4 that q_j has an extension: $Ae_{\varkappa_{i_0}}f_{i_0} \to Ae_{\varkappa_j}f_j$ and by Lemma 4.4 of [11] M is decomposed into $Ae_{\varkappa_j}f_j$, $j \neq i_0$ and a homomorphic image of $Ae_{\varkappa_{i_0}}$.

If $Ae_{\varkappa_{i_0}}f_{i_0}$ is uni-serial, we have also the same conclusion because $Ae_{\varkappa_{i_0}}f_{i_0}$ is injective relative to $N^{l_{i_0}}$.

2. S is not simple. Let T be a simple submodule of S. Then similarly as before we can take elements u_i and v_i of $Ne_{\varkappa_i}$ and obtain with respect to T the following exact sequences:

$$0 \to N^{m_i}u_i + N^{n_i}v_i \to Ae_{\varkappa_i} \to Ae_{\varkappa_i}f_i \to 0,$$
$$0 \to N^{m_i}u_i + N^{n_i}v_i \to N^{m_i-1}u_i + N^{n_i}v_i \to p_i(T) \to 0,$$

where we shall only consider i such that $p_i(T) \neq 0$ and $p_1(T) = T$, and for $i > k$ we also have

$$0 \to N^{l_i}e_{\varkappa_i} \to Ae_{\varkappa_i} \to Ae_{\varkappa_i}f_i \to 0.$$

If $Ae_{\varkappa_1}f_1$ is not a minimum element of $\{Ae_{\varkappa_i}f_i\}$ with respect to T, then choose a minimum element $Ae_{\varkappa_i}f_{i_0}$ (cf. the proof of case 1) and decompose $\sum_{i \neq i_0} Ae_{\varkappa_i}f_i$ into indecomposable summands, each of which, similarly as before, is assumed to be a homomorphic image of $Ae_{\varkappa_i}$, and say, can be denoted by $Ae_{\varkappa_i}f_i'$. In this case $Ae_{\varkappa_{i_0}}f_{i_0} \cap \sum_{i \neq i_0} Ae_{\varkappa_i}f_i' = p_{i_0}(S)$ and $m_1 > m_{i_0}$ or $m_1 = m_{i_0}$, $n_1 < n_{i_0}$.

If $Ae_{\varkappa_{i_0}}f_{i_0}$ is not a minimum element of $\{Ae_{\varkappa_i}f_i',\ i \neq i_0$, and $Ae_{\varkappa_{i_0}}f_{i_0}\}$ with respect to $P_{i_0}(T)$, then we shall apply the same process for $\{Ae_{\varkappa_{i_0}}f_{i_0}$ and $Ae_{\varkappa_i}f_i',\ i \neq i_0\}$. From the condition that $m_1 > m_{i_0}$ or $m_1 = m_{i_0}$, $n_1 < n_{i_0}$ it will be seen that by finite applications of this process we obtain a system of generators, denoting them again $e_{\varkappa_1}f_1,\ e_{\varkappa_2}f_2, \ldots, e_{\varkappa_s}f_s$, of M such that $Ae_{\varkappa_1}f_1$ is a minimum element of $\{Ae_{\varkappa_i}f_i\}$ with respect to a simple submodule T of $Ae_{\varkappa_1}f_1 \cap \sum_{i \geq 2} Ae_{\varkappa_i}f_i$ and $\sum_{i \geq 2} Ae_{\varkappa_i}f_i = Ae_{\varkappa_2}f_2 \oplus Ae_{\varkappa_3}f_3 \oplus \cdots \oplus Ae_{\varkappa_s}f_s$. Then it follows from Proposition 3.3 and 3.4 that projections $p_i\colon T \to Ae_{\varkappa_i}f_i$ have extensions $P_i\colon Ae_{\varkappa_1}f_1 \to Ae_{\varkappa_i}f_i$. Let us denote by $Ae_{\varkappa_1}f_1'$ the left A-module consisting all elements $x - \sum_{i \geq 2} P_i(x)$, where the sum is taken over i with P_i is not zero, $x \in Ae_{\varkappa_1}f_1$. Then clearly $M = Ae_{\varkappa_1}f_1' + \sum_{i \geq 2} Ae_{\varkappa_i}f_i$, and $Ae_{\varkappa_1}f_1' \cap \sum_{i \geq 2} Ae_{\varkappa_i}f_i$ is simple. Hence we arrive at the same situation as the beginning of case 1, and therefore the latter proof is deduced from that of case 1.

Thus we obtain

THEOREM 4.3. *An algebra A is of cyclic-cocyclic type if and only if the following conditions are satisfied:*

I. Each of $Ne_\varkappa$ and $e_\lambda N$ is either uni-serial (i.e. its composition series is unique) or a sum (not necessarily direct sum) of two uni-serial ideals.

II. In case $Ne_\varkappa$ is a sum of two uni-serial ideals, if $e_\mu Ne_\varkappa \not\equiv 0 \bmod J$ for a simple subideal J of $Ne_\varkappa$, then $e_\mu A$ is uni-serial. And the same holds for a right ideal $e_\lambda N$ which is a sum of two uni-serial ideals.

Now we can prove

PROPOSITION 4.4.[2]) *Let A be a finite dimensional algebra with unit element. In order A is of SRCRT it is necessary and sufficient that the following conditions are satisfied*:

I. A is a left generalized uni-serial algebra.

II. Each $e_\varkappa N$ is either uni-serial or a direct sum of two uni-serial ideals.

PROOF. Suppose A is of SRCRT. It is clear that the condition I is satisfied for A. Since A is also of cyclic-cocyclic type, $e_\varkappa N$ is either uni-serial or a sum of two uni-serial ideals. In the latter case, if $e_\varkappa N$ is not a direct sum of two uni-serial ideals, $e_\varkappa N$ has a unique minimal subideal. Hence $e_\varkappa N$ is directly indecomposable, but the top Loewy constituent is not simple. Therefore the condition II is satisfied by SRCRT-algebras.

Next, suppose that the conditions I and II are satisfied. Then A is of cyclic-cocyclic type and from the proof of Theorem 4.1 we have only to prove that every interlacing of at least two uni-serial right A-modules is decomposed. But this follows from condition I, for by duality every uni-serial right A-module is injective relative to a power of the radical.

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[2]) This proposition is equivalent to Theorem 5.5 in [11] and is obtained independently by T. YOSHII [14]. But in his proofs there are some mistakes and gaps, for example Lemma 2 in [14] is incorrect.

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On solutions of the Helmholtz equation in exterior domains

By

PHILIP HARTMAN and CALVIN WILCOX

1. Introduction. This paper deals with functions V that solve the Helmholtz equation $\Delta V + V = 0$ on an exterior domain in Euclidean space $E = E_p$ of dimension $p \geqq 2$ and satisfy the boundedness condition

$$\limsup_{r \to \infty} r^{-1} \int_{\varrho \leqq |x| \leqq r} |V(x)|^2 \, dx < \infty,$$

where $x = (x_1, \ldots, x_p) \in E$, $|x| = (x_1^2 + \cdots + x_p^2)^{\frac{1}{2}}$, and dx is the volume element in E.

The following notation and terminology is used. A domain is an open connected subset of E. A domain D is called an exterior domain if it contains a region $R_\varrho = \{x\colon |x| \geqq \varrho \geqq 0\}$. The letter D below always represents an exterior domain.

For $x \in E$, write $r = |x|$ and, if $x \neq 0$, $u = x/r$. Let $U = \{u\colon |u| = 1\}$ be the surface of the unit sphere in E with surface element du, so that $dx = r^{p-1} dr\, du$. Let $\omega_p = \int_U du$ be the total surface area of U. $L^2(U)$ denotes the Hilbert space of functions $F(u)$ that are square-integrable with respect to du, with inner product and norm

$$(F, G) = \int_U F(u)\, G^*(u)\, du \quad \text{and} \quad ||F|| = (F, F)^{\frac{1}{2}} \geqq 0,$$

where G^* is the complex conjugate of G. Below, l.i.m. always refers to a limit in the space $L^2(U)$.

Functions $V(x)$ on a domain D [or $F(u)$ on U] will be considered identical if they differ only on zero sets with respect to dx [or du]. Correspondingly, if $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_p^2$ is the (Euclidean) Laplacian on E, a complex-valued function $V = V(x)$ on D can be said to solve the Helmholtz wave equation

$$\Delta V + V = 0 \tag{1.1}$$

on D if it is a solution of (1.1) in the usual sense (i.e., of class C^2 on D) or if

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it is a solution in the weak L^2-sense (i.e., if it is locally of class L^2 and satisfies

$$\int_D V(x)\,[\Delta w(x) + w(x)]\,dx = 0$$

for all functions $w(x)$ of class C^∞ on D with compact support); cf., e.g., [*16*, p. 136]. Thus if $V(x)$ solves (1.1) on D, then its real and imaginary parts are (real) analytic on D.

The following notation will be convenient:

$$W(D) = \{V\colon V \text{ solves the Helmholtz equation on } D\},$$
$$W = \{V\colon V \in W(D) \text{ for some exterior domain } D\}.$$

By the Herglotz (boundedness) condition will be meant

$$\limsup_{r\to\infty} r^{-1} \int_{\varrho\le|x|\le r} |V(x)|^2\,dx < \infty \tag{1.2}$$

for some $\varrho \ge 0$. Let

$$W_H(D) = \{V\colon V \in W(D) \text{ and satisfies the Herglotz condition}\},$$
$$W_H = \{V\colon V \in W_H(D) \text{ for some } D\}.$$

Elements V of W_H will be called Herglotz wave functions.

This paper is concerned with a study of W_H. A result of Hartman [*7*, p. 254] shows that a $V \in W_H$ is determined by its behavior at infinity which is described by an element $[K^+(u), K^-(u)]$ of $L^2(U)\times L^2(U)$. In Section 2, it is shown that "lim sup" in (1.2) can be replaced by "lim" and this limit serves as a squared norm $\|V\|_H^2$ which makes W_H a pre-Hilbert space (but not a Hilbert space). Furthermore, the correspondence $V \leftrightarrow [K^+, K^-]$ is an isomorphism between W_H and a certain dense linear manifold W_H' of $L^2(U)\times L^2(U)$.

By using a theorem due to John [*8*] (cf. Wilcox [*21*]), it is shown in Section 3 that W_H is the direct sum of its (non-orthogonal) linear manifolds $W_H(E)$ and W_S, where $W_H(E)$ is the set of entire Herglotz wave functions and W_S is the set of wave functions satisfying the Sommerfeld radiation condition. Thus a further study of W_H is reduced to a study of $W_H(E)$ and of W_S.

Sections 4 and 5 are concerned with $W_H(E)$. It turns out that under the norm $\|V\|_H$, $W_H(E)$ is a Hilbert space. For $V \in W_H(E)$, $K^+(u) = K^-(-u)$ and the correspondence $V \leftrightarrow 2^{\frac{1}{2}} K^+(u)$ is an isomorphism between $W_H(E)$ and $L^2(U)$. An integral representation for V in terms of K^+ is given. Another explicit isomorphism between $W_H(E)$ and $L^2(U)$, suggested to us by a paper of Owens [*13*], is given in Section 5.

In Section 6, several characterizations of elements $V \in W_S$ are given. In particular, there is given a short direct proof of a theorem of Müller [*11*]. Section 7 deals with representation formulae, involving contour integrals, for elements of W_S.

Finally, Section 8 deals with a description of the manifold W_H' and a related manifold in $L^2(U)\times L^2(U)$.

Much of the analysis is based on expansions of functions in spherical harmonics. The space of spherical harmonics of degree n has dimension

$$(1.3)\quad N \equiv N(p;n) = (n+p-3)!\,(2n+p-2)/n!\,(p-2)! = O(n^{p-2})$$

for fixed $p \geqq 2$; [4, p. 237]. If $K_{nj}(u)$, $j=1,\ldots,N$, is an orthonormal basis for this space, then the completeness in $L^2(U)$ of the spherical harmonics implies that every $V \in W$ has an expansion

$$(1.4)\quad r^{\frac{1}{2}(p-1)}\,V(ru) \sim (\tfrac{1}{2}\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n [a_{nj}\,H_\mu^1(r) + b_{nj}\,H_\mu^2(r)]\,K_{nj}(u),$$

to be interpreted in the sense of l.i.m., for fixed $r \geqq \varrho$ for some $\varrho \geqq 0$. Here H_μ^1, H_μ^2 are Hankel functions of order

$$(1.5)\quad \mu \equiv \mu(n,p) = n + \tfrac{1}{2}(p-2),$$

and a_{nj}, b_{nj} are constants. The form of the coefficient of K_{nj} is a consequence of (1.1). Actually, if $V \in W_H(D)$ and $D > R_\varrho$, then the series in (1.4) is uniformly convergent to $r^{\frac{1}{2}(p-1)}V(ru)$ on compact subsets of R_ϱ, but we shall not need this fact. (A proof for the case $V \in W_H(E)$ is included in the proof of Theorem 5.2.)

2. Herglotz wave functions. The correspondence between Herglotz wave functions $V \in W_H$ and certain elements $[K^+(u), K^-(u)]$ of $L^2(U) \times L^2(U)$ can be described as follows:

THEOREM 2.1a. *Let V be a Herglotz wave function*, i.e., *let $V \in W_H$. Then there exist functions K^+, K^- in $L^2(U)$ satisfying*

$$(2.1\pm)\quad K^{\pm}(u) = \underset{r\to\infty}{\text{l.i.m.}}\,(2r)^{-1} \int_\varrho^r e^{\mp i(t+\frac{1}{2}\pi)}\,(\pm it)^{\frac{1}{2}(p-1)}\,[V_r(tu) \pm i\,V(tu)]\,dt,$$

where $V_r = \partial V/\partial r$.

The ordered pair $[K^+, K^-]$ will be called the far field pattern of V. Actually, Theorem 2.1a can be strengthened as follows:

THEOREM 2.1b. *Let $V \in W_H$. Then there exist unique functions K^+, K^- in $L^2(U)$ satisfying, as $r \to \infty$,*

$$(2.2\pm)\quad r^{-1} \int_\varrho^r \left|\left| e^{\mp i(t+\frac{1}{2}\pi)}\,(\pm it)^{\frac{1}{2}(p-1)}\,[V_r(tu) \pm i\,V(tu)] - 2K^{\pm}(u)\right|\right|^2 dt \to 0,$$

and $[K^+, K^-]$ is the far field pattern of V.

That (2.2±) implies (2.1±) is clear from the fact that if $F(x)$ is defined on a domain $D > R_\varrho$ and is of class L^2 on compact subsets of D, then

$$(2.3)\quad \left| r^{-1} \int_\varrho^r F(tu)\,dt \right|^2 \leqq r^{-1} \int_\varrho^r ||F(tu)||^2\,dt.$$

For SCHWARZ's inequality gives

$$\left| r^{-1} \int_\varrho^r F(tu)\,dt \right|^2 \leqq r^{-1} \int_\varrho^r |F(tu)|^2\,dt.$$

If this inequality is integrated with respect to u over U, an application of FUBINI's theorem to the resulting integral on the right gives (2.3).

Theorem 2.1b is equivalent to the following theorem which was proved by HARTMAN [7, p. 254]:

THEOREM 2.1c. *Let $V \in W_H$. Then there exist unique functions K^+, K^- in $L^2(U)$ with the property that if*

$$X(r\,u) = i^{\frac{1}{2}(1-p)}\,K^+(u)\,e^{ir} + i^{\frac{1}{2}(p-1)}\,K^-(u)\,e^{-ir}, \tag{2.4}$$

then

$$r^{-1}\int_\varrho^r ||\,t^{\frac{1}{2}(p-1)}\,V(t\,u) - X(t\,u)||^2\,dt \to 0 \quad as \quad r\to\infty; \tag{2.5}$$

$$r^{-1}\int_\varrho^r ||\,t^{\frac{1}{2}(p-1)}\,V_r(t\,u) - X_r(t\,u)||^2\,dt \to 0 \quad as \quad r\to\infty. \tag{2.6}$$

The relations (2.2±) follow from (2.5), (2.6) by virtue of the identity

$$e^{\mp i\,(t+\frac{1}{2}\pi)}\,(\pm i)^{\frac{1}{2}(p-1)}\,[X_r(t\,u) \pm i\,X(t\,u)] = 2K^\pm(u)$$

and the inequality

$$||F+G||^2 \leqq 2(||F||^2 + ||G||^2) \quad \text{for} \quad F, G \in L^2(U)$$

applied to

$$F(u) = e^{\mp i\,(t+\frac{1}{2}\pi)}\,(\pm i)^{\frac{1}{2}(p-1)}\,[t^{\frac{1}{2}(p-1)}\,V_r(t\,u) - X_r(t\,u)],$$

$$G(u) = \pm i\,e^{\pm i\,(t+\frac{1}{2}\pi)}\,(\pm i)^{\frac{1}{2}(p-1)}\,[t^{\frac{1}{2}(p-1)}\,V(t\,u) - X(t\,u)].$$

In a similar manner (2.5), (2.6) can be deduced from (2.2±).

In the proof of Theorem 2.1c in [7], it was noted that

$$V \in W_H \Rightarrow \sum_{n=0}^{\infty}\sum_{j=1}^{N}(|\,a_{nj}|^2 + |\,b_{nj}|^2) < \infty, \tag{2.7}$$

where a_{nj}, b_{nj} are the constants in (1.4). This remark goes back to MAGNUS [9] in the case $V \in W_H(E)$, where $a_{nj} = b_{nj}$. But his proof carries over to the case $V \in W_H$ as follows. By FUBINI's theorem,

$$r^{-1}\int_{\varrho\leq|x|\leq r} |V(x)|^2\,dx = r^{-1}\int_\varrho^r t^{p-1}\,||V(t\,u)||^2\,dt \tag{2.8}$$

which, by (1.4) and PARSEVAL's relation, is equal to

$$r^{-1}\int_\varrho^r \tfrac{1}{2}\pi\,t\sum_{n=0}^{\infty}\sum_{j=1}^{N}|\,a_{nj}\,H_\mu^1(t) + b_{nj}\,H_\mu^2(t)|^2\,dt. \tag{2.9}$$

Standard asymptotic formulae [17, p. 197] for $H_\mu^1(t)$, $H_\mu^2(t)$ for large t show that

$$r^{-1}\int_\varrho^r \tfrac{1}{2}\,\pi\,t\,|\,a_{nj}\,H_\mu^1(t) + b_{nj}\,H_\mu^2(t)|^2\,dt \to |\,a_{nj}|^2 + |b_{nj}|^2 \quad \text{as} \quad r\to\infty. \tag{2.10}$$

It follows from (1.2), from the equality of (2.8) and (2.9), and from (2.10) that the partial sums of the series in (2.7) are bounded from above by the limit superior in (1.2). This implies (2.7).

The functions $K^{\pm}$ in the proof of Theorem 2.1c are defined to be

$$(2.11)\quad K^{+}(u) \sim \sum_{n=0}^{\infty} \sum_{j=1}^{N} a_{nj} K_{nj}(u), \qquad K^{-}(u) \sim \sum_{n=0}^{\infty} \sum_{j=1}^{N} (-1)^n b_{nj} K_{nj}(u),$$

which is legitimate by (2.7) and the Fischer-Riesz theorem.

The next theorem states that it is possible to define, in a natural way, a scalar product $(V^1, V^2)_H$ for elements $V^1, V^2 \in W_H$ which makes W_H into a pre-Hilbert space. Furthermore, the map $T: W_H \to L^2(U) \times L^2(U)$ defined by

$$(2.12)\qquad TV = [K^{+}, K^{-}] \quad \text{for} \quad V \in W_H,$$

where $[K^{+}, K^{-}]$ is the far field pattern of V is an isomorphism between W_H and the range $W_H' = T(W_H)$ of T. The linear manifold W_H' of $L^2(U) \times L^2(U)$ will be investigated in Section 8 below. It is shown there that W_H' is dense in (but not equal to) $L^2(U) \times L^2(U)$; so that W_H is a pre-Hilbert space, but not a Hilbert space. Also, the linear manifold $W_H(D)$ of W_H, for a fixed exterior domain D, is dense in W_H if and only if $D \neq E$. For $r^{\frac{1}{2}(2-p)} H_{\mu}^{m}(r) K_{nj}(u)$, where $m = 1, 2$, are elements of $W_H(D)$ if D does not contain $x = 0$. As will be clear from Sections 4 and 6, $W_H(D)$ is complete if and only if $D = E$.

THEOREM 2.2. *Let $V \in W$. Then the limit*

$$(2.13)\qquad \|V\|_H^2 = \lim_{r \to \infty} r^{-1} \int_{\varrho \leq |x| \leq r} |V(x)|^2 \, dx \quad \textit{exists}$$

for every fixed sufficiently large ϱ if and only if $V \in W_H$. If $V \in W_H$ and $[K^{+}, K^{-}]$ is its far field pattern, then

$$(2.14)\qquad \|V\|_H^2 = \|K^{+}\|^2 + \|K^{-}\|^2.$$

The usual polarization of (2.13) and (2.14) gives the following:

COROLLARY 2.1. *Let $V^j \in W_H$ have the far field pattern $[K_j^{+}, K_j^{-}]$ for $j = 1, 2$. Then*

$$(V^1, V^2)_H = \lim_{r \to \infty} r^{-1} \int_{\varrho \leq |x| \leq r} V^1(x) \, V^{2*}(x) \, dx \quad \textit{exists}$$

for every large ϱ and satisfies

$$(V^1, V^2)_H = (K_1^{+}, K_2^{+}) + (K_1^{-}, K_2^{-}).$$

Theorem 2.2 and its Corollary 2.1 will be deduced from (2.5). Similar arguments applied to (2.6) lead to

COROLLARY 2.2. *Under the hypotheses of Corollary 2.1,*

$$\lim_{r \to \infty} r^{-1} \int_{\varrho \leq |x| \leq r} V_r^1(x) \, V_r^{2*}(x) \, dx \quad \textit{exists}$$

for every large fixed ϱ and equals $(V^1, V^2)_H$.

A norm similar to $||V||_H$, namely,

$$||V||_G^2 = \lim_{r\to\infty} \tfrac{1}{2} r^{p-1}\{||V(ru)||^2 + ||V_r(ru)||^2\} \tag{2.15}$$

had been defined by GARABEDIAN [5]. He did not discuss the subset W_G of elements $V \in W$ for which $||V||_G$ exists. It is clear that if $||V||_G$ exists, then $||V||_H$ exists (and $||V||_G = ||V||_H$), so that $W_G \subset W_H$. It will be shown in Section 4 below that

$$W_G \neq W_H; \text{ in fact, } W_H(E) \not\subset W_G. \tag{2.16}$$

PROOF OF THEOREM 2.2. The condition $V \in W_H$ is obviously necessary for the existence of $||V||_H$. In order to show that it is sufficient, let $V \in W_H$ and let $[K^+, K^-]$ be its far field pattern. The definition (2.4) of $X(ru)$ shows that

$$||X(tu)||^2 = ||K^+||^2 + ||K^-||^2 + e^{2it} i^{1-p}(K^+, K^-) + e^{-2it} i^{p-1}(K^-, K^+);$$

hence

$$\lim_{r\to\infty} r^{-1} \int_\varrho^r ||X(tu)||^2\, dt = ||K^+||^2 + ||K^-||^2. \tag{2.17}$$

If, $F, G \in L^2(U)$, then the inequality

$$|\,||F||^2 - ||G||^2| \leqq \{||F|| + ||G||\}\,||F - G|| \tag{2.18}$$

follows from the identity $||F||^2 - ||G||^2 = (F, F-G) + (F-G, G)$ and SCHWARZ's inequality. An integration of the inequality (2.18) with respect to t, where $F = t^{\frac{1}{2}(p-1)} V(tu)$ and $G = X(tu)$, implies that

$$\left| r^{-1} \int_\varrho^r ||V(tu)||^2\, t^{p-1}\, dt - r^{-1} \int_\varrho^r ||X(tu)||^2\, dt \right|$$

is majorized by

$$r^{-1} \int_\varrho^r \{||V(tu)||\, t^{\frac{1}{2}(p-1)} + ||X(tu)||\}\, ||V(tu)\, t^{\frac{1}{2}(p-1)} - X(tu)||\, dt,$$

which does not exceed

$$\left[2r^{-1} \int_\varrho^r (||V(tu)||^2\, t^{p-1} + ||X(tu)||^2)\, dt\right]^{\frac{1}{2}} \left[r^{-1} \int_\varrho^r ||V(tu)\, t^{\frac{1}{2}(p-1)} - X(tu)||^2\, dt\right]^{\frac{1}{2}}.$$

The assertions (2.13), (2.14) follow from (2.8), (2.17), (1.2) and (2.5).

3. A decomposition theorem. The Sommerfeld radiation condition is

$$\lim_{r\to\infty} r^{\frac{1}{2}(p-1)}[V_r(ru) - i\, V(ru)] = 0 \quad \text{uniformly for } u \in U. \tag{3.1}$$

Introduce the notation

$$W_S(D) = \{V:\ V \in W(D) \text{ and satisfies the Sommerfeld condition}\},$$
$$W_S = \{V:\ V \in W_S(D) \text{ for some exterior domain } D\}.$$

It is known that (3.1) is implied by weaker forms of the radiation condition; e.g., if the uniform limit in (3.1) is replaced by the ordinary limit (for every fixed u) (MAGNUS [9]) or by l.i.m. (WILCOX [20]). Also (3.1) implies that $r^{\frac{1}{2}(p-1)} V(ru)$ is uniformly bounded in u for large r; in particular, $W_S(D) \subset W_H$; cf. MAGNUS [9], ATKINSON [1], WILCOX [20].

In WILCOX [21], there was introduced a radiation condition, very different in appearance from (3.1), namely,

$$V_1(x) \equiv 0, \tag{3.2}$$

where

$$V_1(x) = (i\pi s^{p-1} H(s)^2/4C^2\omega_p) \int_U \partial/\partial r [V(x + s u')/H(s)]\, du', \tag{3.3}$$

$V \in W(D)$, C is the normalizing constant $\pi(2\pi)^{\frac{1}{2}(p-1)}/\omega_p$ and

$$H(r) = C r^{-\frac{1}{2}(p-2)} H^1_{\frac{1}{2}(p-2)}(r). \tag{3.4}$$

$V_1(x)$ is defined for all x if s is fixed and greater than a sufficiently large constant $s_0(x)$ and $V_1(x)$ is independent of s. Straightforward differentiations show that $V_1 \in W(E)$. The function V_1 arises, for example, in the consideration of the spherical mean

$$\tilde{V}(x, s) = \omega_p^{-1} \int_U V(x + s u')\, du'$$

defined for all x and $s > s_0(x)$. From (1.1), it is readily verified that

$$\tilde{V}_{ss} + (p-1) s^{-1} \tilde{V}_s + \tilde{V} = 0, \qquad s > s_0(x),$$

so that there exist unique functions $V_1(x)$, $V_2(x)$ satisfying

$$\tilde{V}(x, s) = V_1(x) H^*(s) + V_2(x) H(s), \qquad s > s_0(x). \tag{3.5}$$

Furthermore $V_1(x)$ is given by (3.3). An analogous formula is valid for $V_2(x)$ and shows that $V_2 \in W(E)$. The function V_1 can be considered to be defined implicitly by (3.5) or explicitly by (3.3). Introduce the notation

$$\widetilde{W}_S(D) = \{V\colon\ V \in W(D) \text{ and } V_1 \equiv 0\},$$

$$\widetilde{W}_S = \{V\colon\ V \in \widetilde{W}(D) \text{ for some exterior domain } D\}.$$

The relation

$$\widetilde{W}_S(D) = W_S(D) \tag{3.6}$$

was proved in WILCOX [21]; another proof is indicated in Section 6 below.

A basic fact needed below is the following theorem which does not involve the Herglotz condition (1.2).

THEOREM 3. *Let $V \in W(D)$. Then V has a unique decomposition*

$$V(x) = V_0(x) + V_1(x) \quad \textit{for} \quad x \in D, \tag{3.7}$$

where $V_0 \in W_S(D)$ and $V_1 \in W(E)$.

This result was first stated by JOHN in [8]. A similar proof which, however, employs spherical means and the radiation condition (3.2) was given by WILCOX [21] for $p=3$. This proof is extendable for arbitrary p, and can be outlined briefly as follows: Let

$$G(x;x') = i\,H(R)/4C\,(2\pi)^{\frac{1}{2}(p-2)}, \qquad R=|x-x'|.$$

By a standard technique involving Green identities, one obtains a decomposition (3.7) for V, where V_1 is given by (3.3) and V_0 by

$$V_0(x) = \int_B \{G\,\partial V/\partial \nu - V\,\partial G/\partial \nu\}\,dB,$$

where B is a closed hypersurface surrounding the boundary of D, but not x. It is clear that $V_0 \in W(D)$. A simple calculation shows that

$$\tilde{V}_0(x,r) = \tilde{V}(x,r) - \tilde{V}_1(x,r) = [V_2(x) - V_1(x)]\,H(r), \qquad r > r_0(x),$$

so that, by (3.5), $V_0 \in \tilde{W}_S(D)$. This gives a decomposition (3.7) of V into functions $V_1 \in W(E)$, $V_0 \in \tilde{W}_S(D)$. It is easily verified that such a decomposition is unique.

In particular, there results the representation formula

$$V(x) = \int_B \{G(R)\,\partial V(x')/\partial \nu - V(x')\,G'(R)\,\partial R/\partial \nu\}\,dB \tag{3.8}$$

for $V \in \tilde{W}_S(D)$. From this and the asymptotic formula for $G(R)$, $G'(R)$ as $R\to\infty$, one deduces

COROLLARY 3.1. *Let $V \in \tilde{W}_S(D)$. Then, as $r\to\infty$,*

$$r^{\frac{1}{2}(p-1)}\,V(r\,u) = i^{\frac{1}{2}(1-p)}\,K(u)\,e^{ir} + O(1/r), \tag{3.9}$$

$$r^{\frac{1}{2}(p-1)}\,V_r(r\,u) = i^{\frac{1}{2}(3-p)}\,K(u)\,e^{ir} + O(1/r), \tag{3.10}$$

uniformly for $u \in U$,

$$K(u) = \tfrac{1}{2}\,i\,(2\pi)^{\frac{1}{2}(1-p)} \int_B \{\partial V(x)/\partial \nu + i\,u\cdot\nu\,V(x)\}\,e^{-i\,x\cdot u}\,dB, \tag{3.11}$$

where B is a closed hypersurface surrounding the boundary of D and a dot denotes a (Euclidean) scalar product, $x\cdot u = x_1u_1 + \cdots + x_pu_p$.

The formula (3.9) shows that $|V(x)| = O(r^{-\frac{1}{2}(p-1)})$, $r\to\infty$, which implies the Herglotz condition (1.2). Also, (3.9) and (3.10) imply (3.1). Thus

COROLLARY 3.2. *If $V \in \tilde{W}_S(D)$, then $V \in W_H(D)$*; i.e., $\tilde{W}_S(D) \subset W_H(D)$. *Also $\tilde{W}_S(D) \subset W_S(D)$.*

Finally, if $V \in W_H$, then $V_1 = V - V_0 \in W_H$ by the last corollary. Hence

COROLLARY 3.3. *$V \in W_H(D)$ if and only if V has a decomposition (3.7), which is necessarily unique, with $V_0 \in \tilde{W}_S(D)$ and $V_1 \in W_H(E)$.*

The rest of the paper is concerned with the study of the classes $W_H(E)$ and $\tilde{W}_S$.

4. Entire Herglotz wave functions. In this and the next section, we consider the class $W_H(E)$ of entire solutions of the Helmholtz equation which satisfy the Herglotz condition.

A function $V \in W_H(D)$ may, of course, have an extension to a function $V' \in W_H(D')$ where $D' > D$ and such an extension is unique, since solutions of $\Delta V + V = 0$ are analytic functions. For the case $D' = E$, we have

LEMMA 4.1. *Let $V \in W_H$ and let $[K^+, K^-]$ be its far field pattern. Then V has an extension $V' \in W_H(E)$ if and only if*

$$K^-(u) = K^+(-u) \quad \text{for} \quad u \in U. \tag{4.1}$$

PROOF. Let V have an extension $V' \in W_H(E)$. Then $[K^+, K^-]$ is also the far field pattern of V', so that V, V' have the same expansion into spherical harmonics. Since V' is regular at $r = 0$, the coefficient of K_{nj} in (1.2) must be finite there; whence $b_{nj} = a_{nj}$. Since $K_{nj}(-u) = (-1)^n K_{nj}(u)$ hold for all spherical harmonics, (4.1) follows. The converse proposition that (4.1) implies the existence of an extension $V' \in W_H(E)$ is a corollary of Theorem 4 below

Lemma 4.1 shows that to each $V \in W_H(E)$, there corresponds a function $K = K^+ \in L^2(U)$ which determines V uniquely. For brevity, we shall call K the far field pattern of V. The next result makes the correspondence explicit and shows that every $K \in L^2(U)$ is the far field pattern of a $V \in W_H(E)$.

THEOREM 4. *If $V \in W_H(E)$ and K is its far field pattern, then*

$$V(x) = (2\pi)^{\frac{1}{2}(1-p)} \int_U e^{i x \cdot u'} K(u')\, du'. \tag{4.2}$$

Conversely, if $K \in L^2(U)$, then (4.2) defines a $V \in W_H(E)$ with far field pattern K.

The first part of this theorem was given by JOHN [8] in the case $p = 2$.

PROOF. We begin by proving the converse. Let $K \in L^2(U)$ and V be defined by (4.2). It is clear from LEBESGUE'S theorem on dominated convergence (with a majorant of class $L(U)$) that $V(x) \in C^\infty$ and that its derivatives can be calculated formally by differentiation under the integral sign. Since $e^{i x \cdot u'} \in W(E)$ for fixed u' with $|u'| = 1$, it follows that $V \in W(E)$.

The addition formula [*10*, p. 258],

$$r^{\frac{1}{2}(p-1)} (2\pi)^{\frac{1}{2}(1-p)} e^{i r u \cdot u'} \sim (2\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^n J_\mu(r) \, [N(p; n)/\omega_p] \, P_n(p; u \cdot u'), \tag{4.3}$$

and the addition formula [*10*, p. 242] for the n-th Legendre polynomial $P_n(p; t)$ of $E = E_p$,

$$[N(p; n)/\omega_p] \, P_n(p; u \cdot u') = \sum_{j=1}^{N} K_{nj}^*(u') \, K_{nj}(u), \tag{4.4}$$

show that

$$r^{\frac{1}{2}(p-1)} (2\pi)^{\frac{1}{2}(1-p)} e^{i r u \cdot u'} \sim (2\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n K_{nj}^*(u') \, J_\mu(r) \, K_{nj}(u). \tag{4.5}$$

The coefficient of $K_{nj}(u)$ in the expansion of $r^{\frac{1}{2}(p-1)}V(ru)$ is, after an application of FUBINI's theorem,

$$(2\pi)^{\frac{1}{2}(1-p)}\int_U K(u')\Big[\int_U r^{\frac{1}{2}(p-1)}e^{ir u\cdot u'}K^*_{nj}(u)\,du\Big]\,du'.$$

This reduces, by virtue of (4.5) to

$$\int_U K(u')\,[i^n(2\pi r)^{\frac{1}{2}}J_\mu(r)\,K^*_{nj}(u')]\,du'.$$

Thus if

$$K(u)\sim\sum_{n=0}^{\infty}\sum_{j=1}^{N}c_{nj}K_{nj}(u), \tag{4.6}$$

then

$$r^{\frac{1}{2}(p-1)}V(ru)\sim(2\pi r)^{\frac{1}{2}}\sum_{n=0}^{\infty}\sum_{j=1}^{N}i^n c_{nj}J_\mu(r)\,K_{nj}(u). \tag{4.7}$$

As shown by MAGNUS [9, pp. 85–86], the condition $\sum\sum|c_{nj}|^2<\infty$ implies that V satisfies the Herglotz condition (1.2); so that $V\in W_H(E)$. That K is the far field pattern of V follows from Theorem 2.1a, (2.11) and $2J_\mu=H^1_\mu+H^2_\mu$.

In order to obtain the first part of Theorem 4, let $V\in W_H(E)$ and K be its far field pattern. By the part of the theorem just proved, the right side of (4.2) defines a function, say, $V'\in W_H(E)$ with a far field pattern K. But then (4.7) holds for $V=V$ or $V=V'$; so that $V\equiv V'$.

Theorems 2.2 and 4 give a complete characterization of $W_H(E)$ and of its image $W'_H=T\big(W_H(E)\big)$ under the map (2.12). The result can be stated as follows:

COROLLARY 4.1. *The correspondence* $2^{\frac{1}{2}}K\leftrightarrow V$ *defined by* (4.2) *and its inverse*

$$K(u)=\operatorname*{l.i.m.}_{r\to\infty}(2r)^{-1}\int_\varrho^r e^{-i(t+\frac{1}{2}\pi)}(it)^{\frac{1}{2}(p-1)}\{V_r(tu)+iV(tu)\}\,dt \tag{4.8}$$

is an isomorphism of the Hilbert space $L^2(U)$ *onto* $W_H(E)$; i.e., *it is linear, one-to-one and preserves scalar products* $(V^1,V^2)_H=(2^{\frac{1}{2}}K^1,2^{\frac{1}{2}}K^2)$.

REMARK. This makes possible the following proof of the last part of (2.16). Let $V\in W_H(E)$ satisfy (4.7). Since (4.7) can be formally differentiated with respect to r, it follows that

$$r^{\frac{1}{2}(p-1)}V_r(ru)\sim\sum_{n=0}^{\infty}\sum_{j=1}^{N}i^n c_{nj}\{[(2\pi r)^{\frac{1}{2}}J_\mu]'-\tfrac{1}{2}(p-1)r^{-1}(2\pi r^{\frac{1}{2}})J_\mu\}K_{nj}(u).$$

Hence, PARSEVAL's relation gives

$$r^{(p-1)}\{\|V(ru)\|^2+\|V_r(ru)\|^2\}=\sum_{n=0}^{\infty}|c_n|^2\alpha(n,r),$$

where

$$|c_n|^2=\sum_{j=1}^{N}|c_{nj}|^2,\qquad \alpha(n,r)=(|(2\pi r)^{\frac{1}{2}}J_\mu(r)|^2+|[(2\pi r)^{\frac{1}{2}}J_\mu(r)]'|^2)\,(1+o(1))$$

and $o(1)\to 0$ uniformly as $n,r\to\infty$.

The asymptotic formulas for $J_\mu(r)$ and its derivative for large r (and fixed μ) shows that $\alpha(n, r) \to 1$ as $r \to \infty$ for fixed n. By Corollary 4.1, $c_0, c_1, \ldots$ is arbitrary except for the requirement that $\sum |c_n|^2 < \infty$. It follows from TOEPLITZ's theorem ([22], p. 40) that the limit $||V||_G^2$ in (2.16) exists for all $V \in W_H(E)$ if and only if the set of functions $\alpha(n, r)$ are uniformly bounded for $r > r_0$ (r_0 is independent of n). But CAUCHY's formula [17, p. 231], $J_\mu(\mu) \sim c_0 \mu^{-\frac{1}{3}}$, as $\mu \to \infty$, where $c_0 > 0$, shows that this is not the case. Thus $W_H(E) \not\subset W_G$.

5. $W_H(E)$, continued. If $V \in W_H(E)$, it is more convenient to write the expansion (4.7) of V in the form

$$r^{\frac{1}{2}(p-1)} V(r u) \sim (2\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^n c_n J_\mu(r) K_n(u), \tag{5.1}$$

where K_n is a normalized ($||K_n|| = 1$) spherical harmonic of order n. Let

$$F(u) \sim \sum_{n=0}^{\infty} i^n c_n K_n(u). \tag{5.2}$$

OWENS [13], [14] has investigated conditions on F, with $p = 2$ and 3, to insure

$$\int_0^r t^{\frac{1}{2}(p-2)} V(t u)\, dt \to (2\pi)^{\frac{1}{2}} F(u), \qquad r \to \infty \tag{5.3}$$

uniformly for $u \in U$. Replacing uniform convergence by convergence in the sense of l.i.m., we prove the following analogue of Theorem 4 and Corollary 4.1.

THEOREM 5.1. *If $V \in W_H(E)$, then*

$$\text{l.i.m.}_{r \to \infty} \int_0^r t^{\frac{1}{2}(p-2)} V(t u)\, dt = (2\pi)^{\frac{1}{2}} F(u) \quad \textit{exists.} \tag{5.4}$$

The corresponding map $W_H(E) \to L^2(U)$ defined by $V \to 2^{\frac{1}{2}} F$ is an isomorphism onto; it is also given by (5.1)–(5.2).

PROOF. It is sufficient to prove (5.4) and the fact that (5.1) implies (5.2). For if this is granted, the other assertions of Theorem 5 follow from Corollary 4.1 and the fact that the map

$$A K(u) = F(u), \tag{5.5}$$

where

$$K(u) \sim \sum_{n=0}^{\infty} c_n K_n(u), \qquad ||K_n|| = 1, \tag{5.6}$$

is a unitary map of $L^2(U)$ onto itself.

Let $V \in W_H(E)$ and satisfy (5.1). Then $\sum |c_n|^2 < \infty$ [cf. (2.7)] and so (5.2) defines a function $F(u)$ of class $L^2(U)$. Since (5.1) can be formally integrated over finite r-intervals, PARSEVAL's relation gives

$$\left\| \int_0^r t^{\frac{1}{2}(p-2)} V(t u)\, dt - (2\pi)^{\frac{1}{2}} F(u) \right\|^2 = 2\pi \sum_{n=0}^{\infty} |c_n \alpha(n, r)|^2, \tag{5.7}$$

where

$$\alpha(n,r)=\int_0^r J_\mu(t)\,dt-1. \tag{5.8}$$

Thus (5.4) is equivalent to

$$\sum_{n=0}^{\infty}|c_n|^2\,\alpha(n,r)\to 0,\qquad r\to\infty. \tag{5.9}$$

Since [17, p. 31]

$$\alpha(n,r)\to 0 \quad \text{as} \quad r\to\infty, \tag{5.10}$$

(5.9) holds if the series is uniformly convergent for $0\leqq r<\infty$. Consequently, Theorem 5.1 follows from $\sum|c_n|^2<\infty$ and the uniform boundedness of $\alpha(n,r)$. The latter fact is implied by

LEMMA 5.1. *There exists a constant C such that*

$$0\leqq\int_0^r J_\mu(t)\,dt\leqq C \quad \textit{for all } r\geqq 0 \textit{ and } \mu\geqq\tfrac{1}{2}. \tag{5.11}$$

The proof of this lemma will be given below.

Since a unitary operator A satisfies $(AF, AG)=(F,G)$ for $F, G\in L^2(U)$, Theorem 4 and the proof of Theorem 5.1 give the following representation for the inverse of the map (5.4) from V to F.

COROLLARY 5.1. *Let $k(x,u')=(2\pi)^{\frac{1}{2}(1-p)}(A\,e^{-ix\cdot u'})^*$, i.e.,*

$$k(x,u')=(2\pi)^{\frac{1}{2}}r^{\frac{1}{2}(2-p)}\omega_p^{-1}\sum_{n=0}^{\infty}J_\mu(r)\,N(p;n)\,P_n(p;u\cdot u'). \tag{5.12}$$

Then if $F(u)\in L^2(U)$, the function

$$V(x)=\int_U k(x,u')\,F(u')\,du' \tag{5.13}$$

is in $W_H(E)$ and satisfies (5.4).

OWENS [13], [14] considered the map $F\to V$ given by (5.13) for $p=2,3$ and for a more restricted class of functions F. In addition, he derived a simplified formula for the kernel (5.12).

Lemma 5.1 can also be used to prove

THEOREM 5.2. *If the expansion* (5.2) *for $F(u)$ converges absolutely at a point $u_0\in U$, then* (5.3) *holds for $u=u_0$; if the expansion* (5.2) *is uniformly absolutely convergent on some subset U_1 of U, then* (5.3) *holds uniformly for $u\in U_1$. In particular, one can choose $U_1=U$ if*

$$\sum_{n=1}^{\infty}|c_n|n^{\frac{1}{2}(p-2)}<\infty. \tag{5.14}$$

In order to verify Theorem 5.2, note that the expansion (5.1) is uniformly absolutely convergent on compact subsets of E. This follows from the estimates $c_n=O(1)$, $J_\mu(r)=(\frac{1}{2}r)^\mu/\Gamma(1+\mu)\{1+o(1)\}$, $\mu\to\infty$ and [10, p. 242]

$$|K_n(u)|\leqq[N(p;n)/\omega_p]^{\frac{1}{2}} \quad \text{if} \quad \|K_n\|=1. \tag{5.15}$$

Hence for any $u \in U$,

$$\left| \int_0^r t^{\frac{1}{2}(p-2)} V(tu)\, dt - (2\pi)^{\frac{1}{2}} F(u) \right| \leqq (2\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} |c_n K_n(u)|\, |\alpha(n,r)|.$$

This makes it clear that the first two statements of Theorem 5.2 follow from (5.10) and Lemma 5.1. The third depends, in addition, on the estimates (5.15) and (1.3).

Theorem 5.2 implies the result of OWENS [*13*] to the effect that if $p=2$ and $F(u)$, given by (5.2), is uniformly Hölder continuous of order $\lambda > \frac{1}{2}$, then (5.3) holds uniformly on U. In fact, $p=2$ means that (5.2) is a trigonometric Fourier series, so that (5.14) holds if, for example, $F(u)$ is uniformly Hölder continuous of order $\lambda > \frac{1}{2}$ or $F(u)$ is of bounded variation and uniformly Hölder continuous of some order $\lambda > 0$; cf. [*22*, pp. 135—137].

A corresponding result announced by OWENS [*14*] for $p=3$, where Hölder continuity is replaced by Lipschitz continuity ($\lambda = 1$) does not seem to be contained in Theorem 5.2. OWENS' proof is incomplete because he seems to assume that if $f(\vartheta)$ is uniformly Lipschitz continuous for $0 \leqq \vartheta \leqq \pi$, then $g(\xi) = f(\arccos \xi)$ is uniformly Lipschitz continuous for $-1 \leqq \xi \leqq 1$. For this reason, the theorem of OWENS will be proved below and extended to arbitrary $p \geqq 3$.

THEOREM 5.3. *Let $p \geqq 3$ and $j = \frac{1}{2}(p-1)$ or $j = \frac{1}{2}p - 1$ according as p is odd or even. Let the function $F(u)$ be defined on U: $|u| = 1$ and have the Laplace series* (5.2). *Assume that $F(u)$ is of class C^{j-1} on U and that the partial derivatives of order $j-1$ are uniformly Lipschitz condition on U. Let $V \in W_H(E)$ have the expansion* (5.1). *Then* (5.3) *holds uniformly on U.*

This will be deduced from the following three lemmas.

LEMMA 5.2. *Let $\tau \geqq 0$ and let $a_1, a_2, \ldots$ be complex numbers satisfying either*

$$\sum_{n=0}^{\infty} |a_n| < \infty \tag{5.16}$$

or

$$\sum_{n=0}^{\infty} a_n \text{ is convergent and } \sum_{k=0}^{\infty} \left| \sum_{n=k}^{\infty} a_n \right|^2 < \infty. \tag{5.17}$$

Then

$$\int_0^R \sum_{n=0}^{\infty} a_n J_{2n+\tau}(r)\, dr \to \sum_{n=0}^{\infty} a_n \quad \text{as} \quad R \to \infty. \tag{5.18}$$

The assertion (5.18) under the assumption (5.16) is clear from Lemma 5.1. The assumption (5.17) will be considered below.

LEMMA 5.3. *Let $p \geqq 3$ and let $j = j(p)$ be $\frac{1}{2}(p-1)$ or $\frac{1}{2}p - 1$ according as p is odd or even. Let $f(t)$ be defined for $-1 \leqq t \leqq 1$ and have the properties that $f(t)\,(1-t^2)^{\frac{1}{2}(p-3)}$ is of class C^{j-1}, the derivatives of order $0, \ldots, j-1$ vanish at $t = \pm 1$, the derivative of order $j-1$ is absolutely continuous and*

$$\left| [f(t)\,(1-t^2)^{\frac{1}{2}(p-3)}]^{(j)} \right| \leqq M(1-t^2)^{-\alpha} \tag{5.19}$$

holds almost everywhere, where M, α are positive constants and α is subject to $\alpha<\frac{3}{4}$ or $\alpha<\frac{1}{4}$ according as p is odd or even. Then the expansion of $f(t)$ in terms of the polynomials $P_n(p;t)$,

$$f(t)\sim(\omega_{p-1}/\omega_p)\sum_{n=0}^{\infty}N(p;n)\,f_n\,P_n(p;t), \tag{5.20}$$

where

$$f_n=\int_{-1}^{1}f(t)\,P_n(p;t)\,(1-t^2)^{\frac{1}{2}(p-3)}\,dt, \tag{5.21}$$

is convergent to 0 at $t=1$ and the remainders

$$S_k=(\omega_{p-1}/\omega_p)\sum_{n=0}^{k}N(p;n)\,f_n=-(\omega_{p-1}/\omega_p)\sum_{n=k+1}^{\infty}N(p;n)\,f_n \tag{5.22}$$

satisfy

$$\sum_{k=0}^{\infty}|S_k|^2<\infty. \tag{5.23}$$

Remark 1. If, in this lemma, p is odd, the speed of convergence of (5.23) can be estimated in terms of M, α, p. (This fact will be used below to obtain the uniformity of (5.3) in Theorem 5.3.)

Remark 2. If p is even, the lemma remains correct if the assumption (5.19) is replaced by

$$\int_{-1}^{1}|g(t)|^2\,(1-t^2)^{-\frac{1}{2}}\,dt<\infty, \tag{5.24}$$

where

$$g(t)=[f(t)\,(1-t^2)^{\frac{1}{2}(p-3)}]^{(j)}. \tag{5.25}$$

Further, suppose that $f=f(t;u)$ depends on parameters u varying on a compact set, say $U\colon |u|=1$, in such a way that the corresponding functions $g=g(t;u)$ in (5.25) satisfy

$$\int_{-1}^{1}|g(t;u^1)-g(t;u^2)|^2\,(1-t^2)^{-\frac{1}{2}}\,dt\to 0\quad\text{as}\quad u^1-u^2\to 0. \tag{5.26}$$

Then the proof of Lemma 5.3 will show that (5.23) holds uniformly in u.

The last lemma to be needed is

Lemma 5.4. *Let the function (5.2) satisfy the conditions of Theorem 5.3. Then the Laplace series in (5.2) is uniformly convergent on U to $F(u)$ and the remainders*

$$S_k=\sum_{n=0}^{k}i^n\,c_n\,K_n(u)-F(u) \tag{5.27}$$

satisfy (5.23) uniformly on U.

In view of Lemma 5.4, applications of the case (5.17) of Lemma 5.2 to the series $\sum a_n$ representing the Laplace series for $F(u)+F(-u)$ and $F(u)-F(-u)$ give Theorem 5.3.

Lemmas 5.1—5.4 will now be proved.

PROOF OF LEMMA 5.1. Let $0=j_0<j_1<\cdots$ be the non-negative zeros of J_μ. We shall show that

$$\left|\int_{j_{n-1}}^{j_n} J_\mu(t)\,dt\right| \geqq \left|\int_{j_n}^{j_{n+1}} J_\mu(t)\,dt\right| \quad \text{for} \quad n=1,2,\ldots. \tag{5.28}$$

From this, the relation

$$\int_0^r J_\mu(t)\,dt=\sum_{n=1}^{M}\int_{j_{n-1}}^{j_n} J_\mu(t)+\int_{j_M}^{r} J_\mu(t)\,dt, \qquad j_M\leqq r<j_{M+1},$$

and the familiar "alternating series" argument, it follows that

$$0\leqq\int_0^r J_\mu(t)\,dt\leqq\int_0^{j_1} J_\mu(t)\,dt.$$

The boundedness of the last integral for $\mu\geqq\frac{1}{2}$ is a consequence of the following known facts:

$$\int_0^\mu J_\mu(t)\,dt\to\tfrac{1}{3} \quad \text{as} \quad \mu\to\infty$$

(see [17, p. 259]); there exist positive constants C_1, C_2 such that $j_1-\mu\sim C_1\mu^{\frac{1}{3}}$ as $\mu\to\infty$ (see [17, p. 521]) and $J_\mu(r)\leqq C_2/\mu^{\frac{1}{3}}$ for $0\leqq r<j_1$ (see [18, p. 161 and 169]). Thus, the proof of Lemma 5.1 will be complete if (5.28) is verified.

The function $y(r)=r^{\frac{1}{2}}J_\mu(r)$ has the same zeros as J_μ and satisfies the differential equation

$$y''+[1+r^{-2}(\tfrac{1}{4}-\mu^2)]\,y=0,$$

where, for $\mu\geqq\frac{1}{2}$, the coefficient of y is non-decreasing. By a suitable application of STURM's second comparison theorem to the differential equations satisfied by

$$u(s)=|y(j_n+s)|, \qquad u(s)=|y(j_n-s)|$$

for $0\leqq s\leqq\min(j_{n+1}-j_n,\ j_n-j_{n-1})$, it is seen that

$$j_{n+1}-j_n\leqq j_n-j_{n-1} \quad \text{and} \quad |y(j_n+s)|\leqq|y(j_n-s)| \quad \text{for} \quad 0\leqq s\leqq j_{n+1}-j_n;$$

cf. [6, p. 531 and p. 538]. Since the factor $r^{\frac{1}{2}}$ in $y=r^{\frac{1}{2}}J_\mu(r)$ is increasing, the inequalities in the last formula line imply (5.28).

PROOF OF LEMMA 5.2. It only remains to consider the case (5.17) of Lemma 5.2. It is clear that the expression on the left of (5.18) can be integrated term-by-term to obtain

$$I(R)\equiv\sum_{n=0}^{\infty} a_n\int_0^R J_\gamma(r)\,dr, \quad \text{where} \quad \gamma=2n+\tau.$$

By ABEL's summation formula,

$$I(R)=\sum_{n=1}^{\infty} S_n\int_0^R(J_\gamma-J_{\gamma-2})\,dr+S_0\int_0^R J_\tau\,dr,$$

where $\gamma(n-1)=\gamma(n)-2$. Since $J_\gamma-J_{\gamma-2}=-2J'_{\gamma-1}$, it follows that

$$I(R)=-2\sum_{n=1}^{\infty} S_n J_{\gamma-1}(R)+S_0\int_0^R J_\tau\,dr.$$

The series on the right is majorized by

$$2\sum_{n=1}^{m}|S_n|\,|J_{\nu-1}(R)| + \Big(\sum_{n=1}^{\infty}|J_{\nu-1}(R)|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=m}^{\infty}|S_n|^2\Big)^{\frac{1}{2}}.$$

The function $\sum|J_{\nu-1}(R)|^2$ is bounded for $0\leq R<\infty$; [17, p. 152]. Hence, it follows from (5.8) and (5.10) that if m is fixed and large, then, for large R, $I(R)$ is close to S_0. This proves the lemma.

Proof of Lemma 5.3. By the first definition of S_k in (5.22),

$$\omega_p S_k/\omega_{p-1} = \int_{-1}^{1} f(t)\sum_{n=0}^{k} N(p;n)\,P_n(p;t)\,(1-t^2)^{\frac{1}{2}(p-3)}\,dt. \tag{5.29}$$

Note that $P_n(p;1)=1$, so that [4, p. 243]

$$(p-2)\,N(p;n)\,P_n(p;t) = 2(n+\nu)\,C_n^{\nu}(t), \qquad \nu=\tfrac{1}{2}(p-2),$$

where $C_n^{\nu}(t)$ are the Gegenbauer (ultraspherical) polynomials. Hence the recursion formula ([4], p. 175) gives

$$(p-2)\sum_{n=0}^{k} N(p;n)\,P_n(p;t) = (1-t)^{-1}\,[(k+2\nu)\,C_k^{\nu} - (k+1)\,C_{k+1}^{\nu}]. \tag{5.30}$$

The differentiation formulae ([4], p. 176)

$$(k+2\nu)\,C_k^{\nu} = C_{k+1}^{\nu\prime} - t\,C_k^{\nu\prime}, \qquad (k+1)\,C_{k+1}^{\nu} = t\,C_{k+1}^{\nu\prime} - C_k^{\nu\prime}$$

show that the right side of (5.30) is $C_{k+1}^{\nu\prime}+C_k^{\nu\prime}$. Thus, by (5.29) and an integration by parts,

$$(p-2)\,\omega_p S_k/\omega_{p-1} = -\int_{-1}^{1}[f(t)\,(1-t^2)^{\frac{1}{2}(p-3)}]'\,[C_{k+1}^{\nu}+C_k^{\nu}]\,dt.$$

The relation $C_n^{\nu\prime}=2\nu C_{n-1}^{\nu+1}$ (cf. [4], p. 176) and $j-1$ repeated integration by parts give

$$S_k = B_p\int_{-1}^{1} g(t)\,[C_{k+j}^{\delta}+C_{k+j-1}^{\delta}]\,dt, \tag{5.31}$$

where B_p is a constant depending only on p, $g(t)$ is defined by (5.25), and $\delta=\frac{1}{2}$ or $\delta=1$ according as p is odd or even.

Consider first the case that p is odd. Thus

$$|g(t)|\leq M(1-t^2)^{-\alpha} \quad \text{for} \quad |t|<1 \tag{5.32}$$

and $0<\alpha<\frac{3}{4}$. Using the usual notation for the standard Legendre polynomials $P_n(t)=C_n^{\frac{1}{2}}(t)$, (5.31) can be written as

$$S_k = B_p[T_{k-1}+T_k], \quad \text{where} \quad T_k=\int_{-1}^{1} g(t)\,P_{k+j}(t)\,dt. \tag{5.33}$$

Let $0<\varepsilon<1$ and divide the integral defining T_k into three parts

$$T_k = \int_{-1}^{-1+\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1} \equiv \alpha_k+\beta_k+\gamma_k. \tag{5.34}$$

From (5.32) and the classical inequality $|P_n(t)| \leqq 2(n\pi)^{-\frac{1}{2}}(1-t^2)^{-\frac{1}{4}}$ (cf. [4], p. 197), it is seen that

$$|\alpha_k|, \quad |\gamma_k| \leqq M' k^{-\frac{1}{2}} \varepsilon^{\frac{3}{4}-\alpha}, \tag{5.35}$$

where M' is a constant depending only on M, α and p. Since the polynomials $(n+\frac{1}{2})^{\frac{1}{2}} P_n(t)$ are orthonormal for $n=0, 1, \ldots$ (with respect to Lebesgue measure) on $|t| \leqq 1$, BESSEL'S inequality implies

$$\sum_{k=0}^{\infty} |\beta_k|^2 (k+j+\tfrac{1}{2}) \leq \int_{-1+\varepsilon}^{1-\varepsilon} |g(t)|^2 \, dt.$$

Using (5.32),

$$2^m \sum_{k=2^m}^{2^{m+1}-1} |\beta_k|^2 \leqq M'' \varepsilon^{1-2\alpha}, \tag{5.36}$$

where $M'' = M''(M, \alpha, p)$ and it is assumed, without loss of generality, that $\alpha > \frac{1}{2}$.

In (5.35) and (5.36), the quantities α_k, β_k, γ_k depend on ε. Let $2^m \leqq k < 2^{m+1}-1$ and choose $\varepsilon = \varepsilon(m)$ to be $2^{-m\eta}$, where $\eta > 0$ is so small that $(2\alpha-1)\eta < 1$. Then

$$\sum_{k=2^m}^{2^{m+1}-1} (|\alpha_k|^2 + |\beta_k|^2 + |\gamma_k|^2) \leqq M'''(2^{-m\eta(\frac{3}{4}-\alpha)} + 2^{-m[1-(2\alpha-1)\eta]}),$$

$M''' = M'''(M, \alpha, p)$. By (5.34), $\sum |T_k|^2 < \infty$ and so, by (5.33), $\sum |S_k|^2 < \infty$. This completes the proof of Lemma 5.3 for odd p (and also the proof of Remark 1 following it).

Consider the case of even p. The polynomial $C_n^{\delta} = C_n^1(t)$ is the Tchebychef polynomial $U_n(t)$ of the second kind; so that $C_n^1(\cos\vartheta) = \sin(n+1)\vartheta / \sin\vartheta$. Thus (5.31) can be written as

$$S_k = -B_p \int_0^{\pi} g(\cos\vartheta) [\sin(k+j+1)\vartheta + \sin(k+j)\vartheta] \, d\vartheta. \tag{5.37}$$

It follows from BESSEL'S inequality that $\sum |S_k|^2 < \infty$ if $g(\cos\vartheta)$ is of class L^2 on $0 \leqq \vartheta \leqq \pi$. This is certainly the case if (5.32) holds with $\alpha < \frac{1}{4}$, so that the proof of Lemma 5.3 is complete. This argument also makes the validity of Remark 2 clear.

PROOF OF LEMMA 5.4. The question of the convergence of a Laplace series (5.2) at a point u will be reduced in the standard way to the question of the convergence of a series (5.20) at $t=1$.

In view of (4.4), the relation (5.2) can be written as

$$F(u) \sim \sum_{n=0}^{\infty} [N(p; n)/\omega_p] \int_U F(u') P_n(p; u \cdot u') \, du'. \tag{5.38}$$

Let u be fixed and let u^0 be a variable point of U satisfying $u \cdot u^0 = 0$; so that U^0: $|u^0| = 1$ is the boundary of a unit $(p-1)$-dimensional sphere. Writing

$u' = tu + (1-t^2)^{\frac{1}{2}} u^0$, the integral in (5.38) becomes

$$\int_{-1}^{1} P_n(p; t)\,(1-t^2)^{\frac{1}{2}(p-3)} \left\{ \int_{U^0} F\left(tu + (1-t^2)^{\frac{1}{2}} u^0\right) du^0 \right\} dt.$$

Hence, the Laplace series (5.2) at u becomes the series (5.20) at $t=1$ where $f(t)$ is to be identified with

$$f_1(t) = (1/\omega_{p-1}) \int_{U^0} F\left(tu + (1-t^2)^{\frac{1}{2}} u^0\right) du^0. \tag{5.39}$$

Instead of applying Lemma 5.3 to $f(t)=f_1(t)$, apply it to the function

$$f(t) - f_1(t) + L(t), \tag{5.40}$$

where $L(t)$ is a linear function in t chosen so that $f(\pm 1)=0$. The analogue of the series (5.20) for $f(t)$ and for $f_1(t)$ differ only in the 0-th and 1-st terms.

If $F(u)$ is of class C^{j-1} on U, it is clear that $f(t)$ is of class C^{j-1} on $-1<t<1$. Furthermore, $d^k f/dt^k = O\left((1-t^2)^{-k+\frac{1}{2}}\right)$, $t \to \pm 1$, for $k=0, \ldots, j-1$. Also, if the derivatives of order $j-1$ of $F(u)$ are uniformly Lipschitz continuous on U, then $d^{j-1} f/dt^{j-1}$ is uniformly Lipschitz continuous on closed subintervals of $-1<t<1$ and $d^j f/dt^j = O\left((1-t^2)^{-j+\frac{1}{2}}\right)$, $t \to \pm 1$. It follows that $f(t)$ $(1-t^2)^{\frac{1}{2}(p-3)}$ has $j-1$ continuous derivatives for $|t| \leqq 1$ vanishing at $t=\pm 1$. Furthermore, the function (5.25) satisfies (5.32) with $\alpha=\frac{1}{2}$ or $\alpha=0$ according as p is odd or even.

Thus the convergence of the Laplace series (5.2) to $F(u)$ and the inequality (5.23) for (5.27) follows from Lemma 5.3. Furthermore, that (5.23) is uniform in u, when p is odd, follows from Remark 1 and the uniformity of the O-estimates of the last paragraph.

The uniformity of (5.23), when p is even, follows from Remark 2. In order to see this, it is sufficient to verify (5.26). Since $g(t; u)$ is uniformly bounded for $|t|<1$ and $|u|=1$, it is enough to verify the analogue of (5.26), say (5.26ε), in which the integral over $(-1, 1)$ is replaced by the integral over $(-1+\varepsilon, 1-\varepsilon)$ for every fixed $\varepsilon>0$. Suppose that coordinates in E are chosen so that $u^1=(1, 0, \ldots, 0)$ and $u^2=(\cos\varphi, \sin\varphi, 0, \ldots, 0)$. If the function (5.39) is called $f_1(t; u)$, then

$$f_1(t; u^1) = (1/\omega_{p-1}) \int_{U^0} F(u)\, du^0, \qquad f_2(t; u^2) = (1/\omega_{p-1}) \int_{U^0} F(\Omega u)\, du^0,$$

where $u = \left(t, (1-t^2)^{\frac{1}{2}} u_2^0, \ldots, (1-t^2)^{\frac{1}{2}} u_n^0\right)$, $\Omega u = \left(t\cos\varphi - (1-t^2)^{\frac{1}{2}} u_2^0 \sin\varphi,\right.$ $\left. t\sin\varphi + (1-t^2)^{\frac{1}{2}} u_2^0 \cos\varphi, (1-t^2)^{\frac{1}{2}} u_3^0, \ldots, (1-t^2)^{\frac{1}{2}} u_n^0\right)$ and $u^0=(u_2^0, \ldots, u_n^0)$. Note that Ω is a rotation of the sphere U which tends to the identity as $u^1-u^2 \to 0$. The sphere U^0 can be covered by a finite number of patches on each of which suitable local coordinates can be introduced. Suppose, for a moment, that F is of class C_j on U. The functions $g(t; u^1)$, $g(t; u^2)$ can then be calculated formally from the last formula line interchanging differentiations and integration. This makes it clear that, for $|t| \leqq 1-\varepsilon$,

$$|g(t; u^1) - g(t; u^2)| \leqq M(\varepsilon) \left\{ |u^1 - u^2| + \sum{}' \int_{U^0} |G(u) - G(\Omega u)|\, du^0 \right\}.$$

where $M(\varepsilon)$ is a constant depending only on ε and $\sum'$ denotes a sum over all partial derivatives G of F of order $k \leqq j$. By SCHWARZ's inequality

$$|g(t; u^1) - g(t; u^2)|^2 \leqq M'(\varepsilon) \left\{ |u^1 - u^2|^2 + \sum' \int_{U^0} |G(u) - G(\Omega u)|^2 \, du^0 \right\}.$$

Hence $du = (1-t^2)^{\frac{1}{2}(p-3)} \, dt \, du^0$ implies that

$$\int_{-1+\varepsilon}^{1-\varepsilon} |g(t; u^1) - g(t; u^2)|^2 (1-t^2)^{\frac{1}{2}(p-3)} \, dt \leqq$$
$$\leqq M''(\varepsilon) \left\{ |u^1 - u^2|^2 + \sum' \int_U |G(u) - G(\Omega u)|^2 \, du \right\}.$$

It is clear that this inequality remains valid if F satisfies the assumptions of Lemma 5.4 (and is not necessarily of class C^j). In this case, the partial derivatives of F of order j are understood to exist only almost everywhere on U. Finally, the analogue (5.26ε) of (5.26) follows from the last inequality, since Ω tends to the identity as $u^1 - u^2 \to 0$ and all of the derivatives G are essentially bounded [hence of class $L^2(U)$]. This proves Lemma 5.4.

6. The class W_S. In this section, we consider the classes of radiation functions introduced in Section 3. To begin, we collect a number of equivalent forms of the radiation conditions.

LEMMA 6.1. *Let $V \in W$. Then the following properties are equivalent:*

(6.1) $V \in \widetilde{W}_S$, *i.e.*, $V_1 = 0$ [*cf.* (3.3) *or* (3.5)],

(6.2) $V \in W_S$, *i.e.*, V *satisfies* (3.1),

(6.3) $V \in W_H$ *and* $K^- = 0$,

(6.4) *In the expansion* (1.4), $b_{nj} = 0$, *i.e.*,

$$r^{\frac{1}{2}(p-1)} V(r u) \sim (\tfrac{1}{2}\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n a_{nj} H_\mu^1(r) K_{nj}(u)$$

for large (fixed) r.

Note that it is not assumed that $V \in W_H$ in (6.4).

PROOF. We show (6.1) $\Rightarrow$ (6.2) $\Rightarrow$ (6.3) $\Rightarrow$ (6.4) $\Rightarrow$ (6.1). The implication (6.1) $\Rightarrow$ (6.2) has been noted in Corollary 3.2. It has already been mentioned after (3.1) that $W_S \subset W_H$. Also $V \in W_S \Rightarrow K^- = 0$, by (2.1). Hence (6.2) $\Rightarrow$ (6.3). The implication (6.3) $\Rightarrow$ (6.4) is obvious. In order to prove (6.4) $\Rightarrow$ (6.1), let $V \in W$ have the property (6.4). By Corollary 3.3, V has a unique decomposition $V = V_0 + V_1$, where $V_0 \in \widetilde{W}_S$, $V_1 \subset W(E)$. Since (6.1) $\Rightarrow$ (6.4), it follows that V_0 has an expansion of the form

$$r^{\frac{1}{2}(p-1)} V_0(r u) \sim (\tfrac{1}{2}\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n c_{nj} H_\mu^1(r) K_{nj}(u).$$

By the argument in the proof of Lemma 1, the entire wave function V_1 (which is not assumed to be in W_H) has an expansion of the form

$$r^{\frac{1}{2}(p-1)} V_1(r u) \sim (2\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{N} i^n d_{nj} J_\mu(r) K_{nj}(u).$$

Addition of these expansions and a comparison with (6.4) gives

$$a_{nj} H_\mu^1 = c_{nj} H_\mu^1 + 2 d_{nj} J_\mu = (c_{nj} + d_{nj}) H_\mu^1 + d_{nj} H_\mu^2,$$

so that $d_{nj}=0$. Hence $V_1=0$

COROLLARY 6.1. *If $V \in W_S$, then K^+ is the function K defined by* (3.11) *in Corollary* 3.1.

This is clear from (2.1+) and (3.9)—(3.10).

This corollary means that each $V \in W_S$ is characterized by a single function $K=K^+ \in L^2(U)$. We shall call K the far field pattern of $V \in W_S$. Introduce the notation

$$H_\varrho = \{K\colon\ K \text{ is the far field pattern of a } V \in W_S(D_\varrho)\},$$

where $D_\varrho = \{x\colon\ |x| > \varrho\}$, and

$$H = \{K\colon\ K \in H_\varrho \text{ for some } \varrho \geqq 0\}.$$

The elements of the class H have been characterized by MÜLLER [*11*; *12*, pp. 93—94] for the case $p=3$ as follows:

THEOREM 6a. *Let $K \in L^2(U)$ possess the expansion* (5.6). *Then $K \in H$ if and only if the harmonic function defined in $|x|<1$ by*

$$K(x) = \sum_{n=0}^{\infty} c_n r^n K_n(u) = \sum_{n=0}^{\infty} c_n K_n(x) \tag{6.5}$$

can be continued to an entire harmonic function satisfying

$$\limsup_{r \to \infty} r^{-1} \log ||K(ru)||^2 < \infty; \tag{6.6}$$

in fact, $K \in H_\varrho$ if and only if (6.5) *can be extended to an entire harmonic function satisfying*

$$\limsup_{r \to \infty} r^{-1} \log ||K(ru)||^2 \leqq \varrho. \tag{6.7}$$

Since $||K(ru)||^2 = \sum |c_n|^2 r^{2n}$, condition (6.7) is equivalent to

$$\limsup_{n \to \infty} 2|n!\, c_n|^{1/n} \leqq \varrho. \tag{6.8}$$

In fact, (6.7) and (6.8) are well known equivalent conditions that the power series representing $||K(ru)||^2$ define an entire function of exponential type $\tau \leqq \varrho$; cf., e.g., [*2*, p. 8 and p. 11].

We now give a short direct proof of MÜLLER's theorem (for $p \geqq 2$) stated in the form

THEOREM 6b. *Let $K \in L^2(U)$ have the expansion* (5.6). *Then $K \in H_\varrho$ if and only if* (6.8) *holds.*

PROOF. We first show $K \in H_\varrho \Rightarrow$ (6.8). The assumption $K \in H_\varrho$ means that there exists a $V \in W_S(D_\varrho)$ with an expansion

$$r^{\frac{1}{2}(p-1)} V(ru) \sim (\tfrac{1}{2}\pi r)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^n c_n H_\mu^1(r) K_n(u), \qquad r > \varrho. \tag{6.9}$$

By PARSEVAL's relation, $r^{p-1}\|V(ru)\|^2=\frac{1}{2}\pi r\sum|c_nH_\mu^1(r)|^2$ for $r>\varrho$; in particular, $c_nH_\mu^1(r)=O(1)$, $n\to\infty$, for every fixed $r>\varrho$. Thus, since [17, p. 197]

$$\pi i H_\mu^1(r)=\Gamma(\mu)\,(2/r)^\mu\,\big(1+o(1)\big),\qquad \mu\to\infty, \tag{6.10}$$

we have, by $\mu=n+\frac{1}{2}(p-2)$, for large n,

$$|n!\,c_n|\leqq C\,(\tfrac{1}{2}r)^n\,\Gamma(n+1)/\Gamma\big(n+\tfrac{1}{2}(p-2)\big),$$

where C is a constant depending only on r (but not n). Thus

$$|n!\,c_n|^{1/n}\leqq (r/2)\,\big(1+o(1)\big),\qquad n\to\infty.$$

This proves (6.8).

To prove (6.8) $\Rightarrow K\in H_\varrho$, we show that the series in (6.9) is convergent and defines a $V\in W_S(D_\varrho)$. Note that if (6.8) holds, then for every $\varrho'>\varrho$, there is an $n_0(\varrho')$ such that

$$|c_n|\leqq(\tfrac{1}{2}\varrho')^n/n!\quad\text{for}\quad n\geqq n_0(\varrho'). \tag{6.11}$$

The $o(1)$ in (6.10) is uniform with respect to r if $0<\varrho'<r<\varrho''<\infty$. If the inequality (6.11) is combined with (6.10) and the estimate for $K_n(u)$ supplied by (5.15) and (1.3), it is seen that there exist constants $C=C(\varrho',\varrho'')$, $n_0=n_0(\varrho',\varrho'')$ such that for $\varrho'<r<\varrho''$,

$$|c_n\,H_\mu^1(r)\,K_n(u)|\leqq C\,(\varrho'/r)^n\,n^{p-2}\quad\text{for}\quad n\geqq n_0(\varrho',\varrho'').$$

Since $\varrho'>\varrho$ is arbitrary, it follows that the series in (6.9) is uniformly convergent on compact subsets of D_ϱ. Hence its sum represents a function of the form $r^{\frac{1}{2}(p-1)}V(ru)$, where $V(x)$ solves the Helmholtz equation on D_ϱ, i.e., $V\in W(D_\varrho)$. Since V satisfies (6.4) in Lemma 6.1, it follows that $V\in W_S(D_\varrho)$. That K is the far field pattern of V is clear from (6.9) and (5.6).

For the purpose of obtaining integral representations for a $V\in W_S$ in terms of its far field pattern K, we examine the possibility of extending $K(u)$ into an entire harmonic function $K(z)$ of the complex variables $(z_1,\dots,z_p)$ which satisfies a certain growth restriction. In this direction, we get the following analogue of MÜLLER's Theorem 6a.

THEOREM 6c. *Let $K\in L^2(U)$ and $K(x)$ the harmonic function defined by (6.5). Then $K\in H_\varrho$ if and only if $K(x)$ can be extended to an entire harmonic function of the complex vector z which satisfies*

$$|K(z)|\leqq M\exp\{\tfrac{1}{2}(\varrho+\varepsilon)\,[a+(a^2-b^2)^{\frac{1}{2}}]^{\frac{1}{2}}\}\quad\textit{for every}\quad\varepsilon>0, \tag{6.12}$$

where $M=M(\varepsilon)$ is a constant, and

$$a=z\cdot z^*=|z_1|^2+\cdots+|z_p|^2,\qquad b=|z\cdot z|=|z_1^2+\cdots+z_p^2|. \tag{6.13}$$

Note that for complex vectors z and w, the dot product $z\cdot w$ is the sum $z_1w_1+\cdots+z_pw_p$, not the usual scalar product $z_1w_1^*+\cdots+z_pw_p^*$.

PROOF. If $P(t)=a_0t^n+a_1t^{n-1}+\cdots$ is a polynomial, write $P(a,b)=b^nP(a/b)=a_0a^n+a_1a^{n-1}b+\cdots$. Then the addition theorem (4.4) implies

$$\sum_{j=1}^{N}K_{nj}(z)\,K_{nj}^*(z')=(N/\omega_p)\,P_n\big(p;\,z\cdot z',\,[(z\cdot z)\,(z',z')]^{\frac{1}{2}}\big).$$

In particular, if $z' = z^*$,

$$\sum_{j=1}^{N} |K_{nj}(z)|^2 = (N/\omega_p) P_n(p; a, b).$$

Since it can be supposed that, say, $K_n = K_{n1}$, it follows that

$$|K_n(z)|^2 \leqq (N/\omega_p) P_n(p; a, b). \tag{6.14}$$

From the integral representation [*10*, p. 245] for $P_n(p; t)$,

$$P_n(p; t) = (\omega_{p-2}/\omega_{p-1}) \int_{-1}^{1} [t + i(1 - t^2)^{\frac{1}{2}} s]^n (1 - s)^{\frac{1}{2}(p-4)} \, ds,$$

we have

$$P_n(p; a, b) = (\omega_{p-2}/\omega_{p-1}) \int_{-1}^{1} [a + i(b^2 - a^2)^{\frac{1}{2}} s]^n (1 - s)^{\frac{1}{2}(p-4)} \, ds.$$

Since [*10*, p. 243]

$$\omega_{p-1} = \omega_{p-2} \int_{-1}^{1} (1 - s)^{\frac{1}{2}(p-4)} \, ds,$$

and, for $|s| \leqq 1$, $0 \leqq b \leqq a$,

$$|a + i(b^2 - a^2)^{\frac{1}{2}}| \leqq a + (a^2 - b^2)^{\frac{1}{2}},$$

we get

$$|P_n(p; a, b)| \leqq [a + (a^2 - b^2)^{\frac{1}{2}}]^n \quad \text{for} \quad 0 \leqq b \leqq a. \tag{6.15}$$

From $N = O(n^{p-2})$, (6.8), (6.14) and (6.15), it follows that there is an $M = M(\varepsilon)$ satisfying

$$|c_n K_n(z)| \leqq M\{\tfrac{1}{2}(\varrho + \varepsilon)[a + (a^2 - b^2)^{\frac{1}{2}}]\}^n / n! \quad \text{for} \quad n = 0, 1, \ldots. \tag{6.16}$$

Consequently, if $K \in H_\varrho$, then $K(z) = \sum c_n K_n(z)$ is convergent for all complex z and defines an entire harmonic function of z satisfying (6.12).

To prove that (6.12) $\Rightarrow K \in H_\varrho$, note that if $z = x$ real, we have $a = b = r$. Hence (6.12) $\Rightarrow |K(ru)| \leqq M \exp[\frac{1}{2}(\varrho + \varepsilon) r]$ for every $\varepsilon > 0$ which, in turn, implies (6.7). Hence $K \in H_\varrho$ by Theorem 6a.

If $V \in W_S(D_\varrho)$ and $K \in H_\varrho$ is its far field pattern, then by Corollary 6.1,

$$K(u) = -\tfrac{1}{2} i (2\pi)^{\frac{1}{2}(1-p)} \int_{B(\varrho')} [V_r(\varrho' u') + i u \cdot u' V(\varrho' u')] e^{-i\varrho' u \cdot u'} \, dB$$

for all real unit vectors u. Here $B(\varrho)$ is the hypersphere $\{x: |x| = \varrho\}$ and $\varrho' > \varrho$. The right side of this equation possesses an obvious extension to an analytic function of p complex variables $z = (z_1, \ldots, z_p)$, namely,

$$K'(z) = -\tfrac{1}{2} i (2\pi)^{\frac{1}{2}(1-p)} \int_{B(\varrho')} [V_r(\varrho' u') + i z \cdot u' V(\varrho' u')] e^{-i\varrho' z \cdot u'} \, dB. \tag{6.17}$$

This function is not identical with the harmonic extension $K(z)$ of $K(u)$; e.g., they do not coincide at the origin $z = (0, \ldots, 0)$. We can prove, however, that $K(z) = K'(z)$ whenever $z_1^2 + \cdots + z_p^2 = 1$.

LEMMA 6.2. *If $K(u)$ is analytic on the real unit hypersphere $u_1^2+\cdots+u_p^2=1$, then any extension of $K(u)$ to an analytic function $K(z)$ on the complex hypersphere $z_1^2+\cdots+z_p^2=1$ is unique.*

PROOF. Consider two extensions $K(z)$, $K'(z)$ of $K(u)$ analytic on the complex hypersphere Σ: $z_1^2+\cdots+z_p^2=1$. Let $z^0\in\Sigma$. Then there is at least one component, say z_p, of z such that $\operatorname{Re} z_p^2>0$. If $z=(z_1, \ldots, z_{p-1}, [1-(z_1^2+\cdots+z_{p-1}^2)]^{\frac{1}{2}})$, and the square root is chosen so that $z_p^0=[1-(z_1^{0\,2}+\cdots+z_{p-1}^{0\,2})]^{\frac{1}{2}}$, then K, K' become analytic functions, say K_1, K_1', of $(z_1, \ldots, z_{p-1})$ on the domain $Q=\{(z_1, \ldots, z_{p-1})\colon \operatorname{Re}[1-(z_1^2+\cdots+z_{p-1}^2)]>0\}$. This domain contains the point $(z_1^0, \ldots, z_{p-1}^0)$ and the real points $(u_1, \ldots, u_{p-1})$ satisfying $u_1^2+\cdots+u_{p-1}^2<1$. For such real points, $K_1=K_1'$. Thus, by a standard theorem on functions of several complex variables (cf. [2, p. 34]), $K_1\equiv K_1'$ on Q. The relation $K_1=K_1'$ at the point $(z_1^0, \ldots, z_{p-1}^0)$ of Q means that $K=K'$ at the point z^0. Since z^0 is an arbitrary point of Σ, the lemma is proved.

7. Integral representation for $V\in W_S$. One such representation,

$$V(r\,u)=\int_{1+i\,0}^{1+i\,\infty} e^{i\,r\,w}\,\varphi(u,w)\,dw, \qquad \text{where} \qquad \varphi=-\,i\sum_{n=0}^{\infty} c_n P_n(w)\,K_n(u),$$

was proved by MÜLLER [11] for the case $p=3$. In this section, we derive two different integral representations which involve the extension $K(z)$ of $K(u)$, rather then the analogue of $\varphi(u, w)$. The first is given by

THEOREM 7.1. *Let $V\in W_S(D_\varrho)$ and $K\in H_\varrho$ its far field pattern. Then*

$$(2\pi)^{\frac{1}{2}}(i\,r)^{\frac{1}{2}(p-2)}\,V(r\,u)=\int_\Gamma e^{i\,r\cos\varphi+i\frac{1}{2}(p-2)\varphi}\,K(e^{i\varphi}u)\,d\varphi, \qquad r>\varrho,$$

where $K(z)$, $z=e^{i\varphi}u$, is the (unique) harmonic extension of $K(u)$ defined in Theorem 6c and Γ is the oriented rectilinear path from $-\frac{1}{2}\pi+i\infty$ to $-\frac{1}{2}\pi$, to $\frac{1}{2}\pi$, to $\frac{1}{2}\pi-i\infty$ in the φ-plane.

PROOF. It is known that [17, p. 178]

$$H_\mu^1(r)=\pi^{-1}e^{-i\mu\pi}\int_\Gamma e^{i\,r\cos\varphi+i\mu\varphi}\,d\varphi.$$

If this is inserted into the expansion (6.9) for $V(ru)$ and the order of summation and integration reversed, we get

$$r^{\frac{1}{2}(p-1)}\,V(r\,u)=(r/2\pi)^{\frac{1}{2}}\,i^{\frac{1}{2}(2-p)}\int_\Gamma e^{i\,r\cos\varphi+i\frac{1}{2}(p-2)\varphi}\sum_{n=0}^{\infty} c_n\,e^{i\,n\,\varphi}\,K_n(u)\,d\varphi, \qquad r>\varrho.$$

This formula is equivalent to the assertion. The term-by-term integration can be justified by LEBESGUE's theorem on dominated convergence. To see this, note that

$$\left|\sum_{n=0}^{m} c_n\,e^{i\,n\,\varphi}\,K_n(u)\right|\leqq\sum_{n=0}^{\infty}|c_n\,e^{i\,n\,\varphi}\,K_n(u)|\leqq M\exp[\tfrac{1}{2}(\varrho+\varepsilon)e^{-\operatorname{Im}\varphi}]$$

by (6.12). Thus, on taking $\varphi=\pm(\frac{1}{2}\pi-is)$ and $s\geqq 0$, we get

$$\left|e^{i\,r\cos\varphi+i\frac{1}{2}(p-1)\varphi}\sum_{n=0}^{m} c_n\,e^{i\,n\,\varphi}\,K_n(u)\right|\leqq M'\exp\{\pm\tfrac{1}{2}(p-2)\,s-\tfrac{1}{2}[r\pm(\varrho+\varepsilon)]\,e^s\},$$

and the last function is integrable on $0 \leqq s < \infty$ for $r > \varrho + \varepsilon$. This gives Theorem 7.1.

Our second integral representation for elements $V \in W_S$ is closely related to the plane wave representation (4.2) for $V \in W_H(E)$. To state it, we introduce a domain of integration Γ_1 on the complex unit hypersphere $\{w\colon w$ complex, $w_1^2 + \cdots + w_p^2 = 1\}$ as follows: Every $u \in U$ can be written as $u = e_1 \cos\vartheta + u^0 \sin\vartheta$, where $0 \leqq \vartheta \leqq \pi$, $e_1 = (1, 0, \ldots, 0)$ and $u^0 \in U^0$, the boundary of the unit sphere in the $p-1$ dimensional space $x_1 = 0$. Moreover,

$$du = \sin^{p-2}\vartheta \, d\vartheta \, du^0, \tag{7.1}$$

where du^0 is the element of area on U^0. Let

$$\Gamma_1 = \{w\colon w = e_1 \cos\vartheta + u^0 \sin\vartheta,\ u^0 \in U^0 \text{ and } \vartheta \in M_1\}, \tag{7.2}$$

where M_1 is the rectilinear contour in the ϑ-plane

$$M_1\colon 0 \text{ to } \tfrac{1}{2}\pi, \text{ to } \tfrac{1}{2}\pi - i\infty. \tag{7.3}$$

On Γ_1, define the "element of volume" as

$$dw = \sin^{p-2}\vartheta \, d\vartheta \, du^0, \qquad \vartheta \in M_1. \tag{7.4}$$

THEOREM 7.2. *Let $V \in W_S(D_\varrho)$ and $K \in H_\varrho$ its far field pattern. Then*

$$V(x) = (2\pi)^{\frac{1}{2}(1-p)} \int_{\Gamma_1} e^{ix\cdot w} K(w)\, dw \quad \text{for} \quad x_1 > \varrho, \tag{7.5}$$

where $K(w)$ is the unique analytic extension of $K(u)$ to the complex hypersphere $w_1^2 + \cdots + w_p^2 = 1$.

Equation (7.5) is to be interpreted to mean

$$V(x) = (2\pi)^{\frac{1}{2}(1-p)} \int_{U^0} \int_{M_1} e^{i(x_1 w_1 + \cdots + x_p w_p)} K(e_1 \cos\vartheta + u^0 \sin\vartheta) \sin^{p-2}\vartheta \, d\vartheta \, du^0, \tag{7.6}$$

where

$$w_1 = \cos\vartheta; \qquad w_2 = u_2^0 \sin\vartheta, \ldots, w_p = u_p^0 \sin\vartheta; \qquad u^0 = (0, u_2^0, \ldots, u_p^0). \tag{7.7}$$

M_1 is defined relative to a fixed coordinate system. Obviously, by rotating the coordinate system, we can obtain a representation (7.5) valid in an arbitrary half-space which does not meet the hypersphere $B(\varrho)\colon r = \varrho$.

The proof of Theorem 7.2 will be broken into several parts. The first is

LEMMA 7.1. *The integral in (7.5) converges for $x_1 > \varrho$ and converges uniformly on any half-space $x_1 \geqq \varrho' > \varrho$.*

PROOF. Only the portion $\vartheta = \frac{1}{2}\pi - is$, $s \geqq 0$, of M_1 needs to be considered. Here $\cos\vartheta = i \sinh s$, $\sin\vartheta = \cosh s$. Thus

$$\left| e^{ix\cdot w} \right| = e^{-\operatorname{Im} x\cdot w} = e^{-x_1 \sinh s}.$$

In the notation of Theorem 6c, for $z = w$,

$$a = |w_1|^2 + \cdots + |w_p|^2 = \sinh^2 s + \cosh^2 s, \qquad b = 1.$$

Hence

$$[a + (a^2 - b^2)^{\frac{1}{2}}]^{\frac{1}{2}} \leqq (2a)^{\frac{1}{2}} = (e^{2s} + e^{-2s})^{\frac{1}{2}} \leqq e^s (1 + \tfrac{1}{2} e^{-4s}),$$

and, by Theorem 6c,

$$|K(w)| \leqq M \exp[\tfrac{1}{2}(\varrho+\varepsilon)(e^s+\tfrac{1}{2}e^{-3s})] \leqq M' \exp[\tfrac{1}{2}(\varrho+\varepsilon)e^s].$$

It follows that, for $s \geqq 0$,

$$(7.8) \quad |e^{ix\cdot w} K(w) \sin^{p-2}\vartheta| \leqq M'' \exp[-\tfrac{1}{2}(x_1-\varrho-\varepsilon)e^s + (p-2)s],$$

which assures the uniform convergence of the integral in (7.5) on any half-space $x_1 \geqq \varrho' > \varrho$.

Next we prove Theorem 7.2 for the particular case $K=1$.

LEMMA 7.2. *For* $x_1 > 0$,

$$\int_{\Gamma_1} e^{ix\cdot w}\, dw = \tfrac{1}{2}(2\pi)^{\frac{1}{2}p} r^{\frac{1}{2}(2-p)} H^1_{\frac{1}{2}(p-2)}(r).$$

PROOF. When $x_2 = \cdots = x_p = 0$, so that $x_1 = r$, the lemma states that

$$\omega_{p-1} \int_{M_1} e^{ir\cos\vartheta} \sin^{p-2}\vartheta\, d\vartheta\, d\vartheta = \tfrac{1}{2}(2\pi)^{\frac{1}{2}p} r^{\frac{1}{2}(p-2)} H^1_{\frac{1}{2}(p-2)}(r), \qquad r>0.$$

This is easily verified by an induction on p. The general case can be reduced to this by a suitable rotation of coordinates. This has been carried out in detail by H. WEYL [*19*, pp. 481—482 and pp. 491—492] in the case $p=3$. His method is easily carried over to the general case $p \geqq 2$; we do not give details here.

The proof of Theorem 7.2 will now be given on the basis of these two lemmas and the integral representation (3.8) for V, taking for B the hypersphere $B(\varrho')$ with radius $\varrho' > \varrho$:

$$(7.9) \qquad V(x) = \int_{B(\varrho')} [V(x')\, G_r(x-x') - V_r(x')\, G(x-x')]\, dB, \qquad r > \varrho'.$$

By the last lemma, the GREEN's function G has the representation

$$G(x-x') = i\, 2^{-2}(2\pi R)^{\frac{1}{2}(2-p)} H^1_{\frac{1}{2}(p-2)}(R) = \pi i (2\pi)^{-p} \int_{\Gamma_1} e^{i(x-x')\cdot w}\, dw$$

in $x_1 - x_1' > 0$. Also, if $x' = r'u'$, then

$$G_{r'}(x-x') = -i(\pi i)(2\pi)^{-p} \int_{\Gamma_1} e^{i(x-x')\cdot w}(u'\cdot w)\, dw, \qquad x_1 - x_1' > 0.$$

Differentiation under the integral sign is easily justified by using FUBINI's theorem and the absolute convergence of the last integral. These expressions for G and $G_{r'}$ can be substituted in (7.9) provided $x_1 - \varrho' > 0$. The order of integrations in the resulting expressions can be reversed because the integrand

$$e^{i(x-x')\cdot w}[i\, V(x')\, u'\cdot w + V_r(x')] \sin^{p-2}\vartheta$$

is dominated by a constant times

$$\exp[-\tfrac{1}{2}(x_1-\varrho')e^s + (p-1)s] \quad \text{for} \quad \vartheta = \tfrac{1}{2}\pi - is, \qquad s \geqq 0.$$

Thus we have, for $x_1 > \varrho'$,

$$V(x) = -\pi i (2\pi)^{-p} \int_{\Gamma_1} e^{ix\cdot w} \int_{B(\varrho')} [V_r(x') + i\, u'\cdot w\, V(x')]\, e^{-ix'\cdot w}\, dB\, dw.$$

Since the function $K'(z)$ in (6.17) satisfies $K'(w) = K(w)$ on the complex unit hypersphere, the relation in (7.5) follows for $x_1 > \varrho'$. As $\varrho' > \varrho$ is arbitrary, Theorem 7.2 is proved.

The estimate (7.8) for the contribution of the non-real $w \in \Gamma_1$ to the integral representation (7.5) implies that, apart from an exponentially small error, $V(x)$ can be obtained directly from its far field pattern $K(u)$, $u \in U$. More precisely, we have

COROLLARY 7.1. *Let* $U_1 = \{u\colon u \in U$ *and* $u_1 > 0\}$. *Then*

$$V(x) = (2\pi)^{\frac{1}{2}(2-p)} \int_{U_1} e^{ix\cdot u} K(u)\, du + O(e^{-\frac{1}{2}(x_1 - \varrho')}) \quad \text{for} \quad x_1 > \varrho',$$

where $\varrho' > \varrho$ *is arbitrary. The constant implicit in the O-term depends on* ϱ', *but not on* x.

8. The class W_H' and $W_\pm$. In the first part of this section, we complete the discussion of the class of Herglotz wave functions by characterizing the class W_H' of far field patterns $[K^+, K^-]$. Our result which is an immediate corollary of Theorem 3 and 4 is as follows

THEOREM 8.1. *The element* $[K^+(u), K^-(u)]$ *of* $L^2(U) \times L^2(U)$ *is a far field pattern of a function* $V \in W_H(D_\varrho)$ *if and only if*

$$K^+(u) - K^-(-u) \in H_\varrho, \tag{8.1}$$

that is, if and only if

$$\limsup_{n\to\infty} 2 \left| n! \left(\sum_{j=1}^{N} |a_{nj} - b_{nj}|^2 \right)^{\frac{1}{2}} \right|^{1/n} \leqq \varrho. \tag{8.2}$$

PROOF. Every $V \in W_H(D_\varrho)$ has a unique decomposition $V = V_0 + V_1$ with $V_0 \in W_S(D_\varrho)$, $V_1 \in W_H(E)$, by Corollary 3.3. Let $K_0 \in H_\varrho$, $K_1 \in L^2(U)$ be the far field pattern of V_0, V_1, respectively. Then

$$K^+(u) = K_0(u) + K_1(u), \qquad K^-(u) = K_1(-u), \tag{8.3}$$

whence

$$K_0(u) = K^+(u) - K^-(-u) \in H_\varrho. \tag{8.4}$$

Conversely, if (8.1) holds, define K_0, K_1 by (8.3). If $V_0 \in W_S(D_\varrho)$, $V_1 \in W_H(E)$ are the corresponding wave functions, then $V = V_0 + V_1$ is an element of $W_H(D_\varrho)$ and has the far field pattern $[K^+, K^-]$. This completes the proof.

A corollary of Theorem 8.1 is the assertion

$$W_H' = \{[K^+, K^-]\colon K^\pm \in L^2(u),\ K^+(u) - K^-(-u) \in H\},$$

(cf. the definition of H before Theorem 6a) or in terms of the expansions (2.11) for $K^\pm \in L^2(U)$,

$$W_H' = \left\{ [K^+, K^-]\colon \limsup_{n\to\infty} \left| n! \left(\sum_{j=1}^{N} |a_{nj} - b_{nj}|^2 \right)^{\frac{1}{2}} \right|^{1/n} < \infty \right\}.$$

In conclusion we consider the question as to when a wave function V can be decomposed into a sum of "outgoing" and "incoming" waves, i.e., $V = V_1 + V_2$ with V_1, $V_2^* \in W_S$. If such a decomposition exists, it is unique,

by Lemma 6.1. A solution to this problem follows at once from the results of Sections 2 and 6. In terms of the notation

$$W_{\pm}(D_\varrho) = \{V:\ V = V_1 + V_2 \quad \text{with} \quad V_1, V_2^* \quad \text{in} \quad W_S(D_\varrho)\},$$
$$W_{\pm} = \{V:\ V \in W_{\pm}(D_\varrho) \quad \text{for some} \quad \varrho > 0\},$$

we state it as follows

THEOREM 8.2. *$W_{\pm} \subset W_H$. Let $V \in W_H$ and $[K^+, K^-]$ be its far field pattern. Then $V \in W_{\pm}$ if and only if $K^+, K^- \in H$.*

PROOF. The assertion $W_{\pm} \subset W_H$ is contained in Corollary 3.2. If $V \in W_{\pm}$ and its corresponding summands V_1, V_2 have far field patterns K_1, K_2, then $K^+ = K_1$, $K^- = K_2$ are in H by Corollary 3.1. Conversely, if $K^- \in H$, let V_1 and V_2^* be the wave functions in W_S with far field patterns K^+, $(K^-)^*$, respectively. Then V, $V_1 + V_2 \in W_H$ and have the same far field pattern $[K^+, K^-]$. Hence $V = V_1 + V_2 \in W_{\pm}$.

Theorem 8.2 implies

COROLLARY 8.1. *Let $V \in W(D_\varrho)$ and have the expansion* (1.4). *Then $V \in W_{\pm}(D_\varrho)$ if and only if*

$$\limsup_{n\to\infty} 2\left[n!\left(\sum_{j=1}^{N} |a_{nj}|^2 + |b_{nj}|^2\right)^{\frac{1}{2}}\right]^{1/n} \leqq \varrho \tag{8.5}$$

or equivalently, for $\varrho' > \varrho$,

$$\limsup_{n\to\infty} 2\left[n!\left(\lim_{r\to\infty} r^{-1} \int_{\varrho' \leq |x| \leq r} \left| \int_U V(ru')\, P_n(p; u\cdot u')\, du' \right|^2 dx\right)^{\frac{1}{2}}\right]^{1/n} \leqq \varrho. \tag{8.6}$$

PROOF. By (2.11) and Theorem 6b, $K^+ \in H_\varrho$ if and only if

$$\limsup_{n\to\infty} 2\left[n!\left(\sum_{j=1}^{N} |a_{nj}|^2\right)^{\frac{1}{2}}\right]^{1/n} \leqq \varrho$$

and a similar remark, in which b_{nj} replaces a_{nj}, is valid for K^-. Since

$$\max\left[\sum_{j=1}^{N} |a_{nj}|^2,\ \sum_{j=1}^{N} |b_{nj}|^2\right] \leqq \sum_{j=1}^{N} \left(|a_{nj}|^2 + |b_{nj}|^2\right) \leqq 2\max\left[\sum_{j=1}^{N} |a_{nj}|^2,\ \sum_{j=1}^{N} |b_{nj}|^2\right],$$

the assertion concerning (8.5) is clear.

We have [*10*, p. 242]

$$r^{\frac{1}{2}(p-1)} \int_U V(ru')\, P_n(p; u\cdot u')\, du' = (\omega_p/N)\,(\tfrac{1}{2}\pi r)^{\frac{1}{2}}\, i^n \sum_{j=1}^{N} [a_{nj} H_\mu^1(r) + b_{nj} H_\mu^2(r)]\, K_{nj}(u),$$

whence

$$\int_U \left| \int_U V(ru')\, P_n(p; u\cdot u')\, du' \right|^2 r^{p-1}\, du = (\omega_p/N)^2 \sum_{j=1}^{N} (\tfrac{1}{2}\pi r)\, |a_{nj} H_\mu^1(r) + b_{nj} H_\mu^2(r)|^2.$$

Thus

$$r^{-1} \int_{\varrho' \leq |x| \leq r} \left| \int_U V(ru')\, P_n(p; u\cdot u')\, du' \right|^2 dx = (\omega_p/N)^2 \sum_{j=1}^{N} r^{-1} \int_{\varrho'}^{r} (\tfrac{1}{2}\pi t)\, |a_{nj} H_\mu^1(t) + b_{nj} H_\mu^2(t)|^2\, dt.$$

In view of the limit relations (2.10) and $(N/\omega_p)^{1/n} = (N(p; n)/\omega_p)^{1/n} \to 1$ as $n \to \infty$, condition (8.6) is equivalent to (8.5).

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Kombinatorische Eigenschaften endlicher Inzidenzstrukturen

Von

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1. Einleitung. Unter einer *Inzidenzstruktur* soll hier — anders als bei Pickert [8, S. 2] — eine Menge von „Punkten" und „Blöcken" (die manchmal irreführende Bezeichnung „Gerade" wird vermieden) mit beliebiger Inzidenzrelation verstanden werden; eine genaue Definition folgt in **2**. Viele in der Literatur bisher untersuchte *endliche* Inzidenzstrukturen lassen sich zwanglos durch gewisse der folgenden Forderungen beschreiben:

(P_i) *Die Anzahl der mit jedem von i beliebigen Punkten inzidenten Blöcke ist von Null verschieden und nur von i abhängig* ($i = 1, 2, \ldots$).

(B_j) *Die Anzahl der mit jedem von j beliebigen Blöcken inzidenten Punkte ist von Null verschieden und nur von j abhängig* ($j = 1, 2, \ldots$).

Bezeichnet b_i die Anzahl der Blöcke durch je i Punkte und p_j die Anzahl der Punkte auf je j Blöcken, so sind z.B. die endlichen projektiven Ebenen gerade die (P_2) und (B_2) mit $b_2 = p_2 = 1$ erfüllenden endlichen Inzidenzstrukturen, und die ausgearteten Ebenen lassen sich aus ihnen etwa durch die zusätzliche Forderung von (B_1) mit $p_1 > 2$ ausscheiden. Die von Witt [11] untersuchten *Steiner-Systeme* (vgl. auch Pickert [8, S. 286]) sind genau die (B_1) und ein (P_n) mit $n > 1 = b_n$ erfüllenden endlichen Inzidenzstrukturen, und die in der Statistik auftretenden *Blockpläne* (Pickert [8, S. 287], dasselbe wie balanced incomplete block design, vgl. z.B. Bose [1]) erfüllen (B_1) und (P_2).

In der vorliegenden Arbeit werden die zwischen den (P_i) und (B_j) bestehenden logischen Abhängigkeiten und daraus sich ergebende Eigenschaften der zugehörigen Inzidenzstrukturen untersucht. Da (P_i) und (B_j) für $i > p_0$ und $j > b_0$ natürlich trivial erfüllt sind (p_0 und b_0 bezeichnen die Anzahlen aller Punkte bzw. aller Blöcke), kann man sich auf die Fälle $1 \leqq i \leqq p_0$ und $1 \leqq j \leqq b_0$ beschränken. Die Frage nach den Abhängigkeiten zwischen diesen (P_i) und (B_j) läßt sich nun bei Erfülltsein von wenigstens einem der Axiome (P_1), (B_1) befriedigend beantworten: man zeigt leicht, daß aus (B_1) und (P_n) immer $(P_1), \ldots, (P_{n-1})$ folgen, und den dualen Sachverhalt (Satz 2). Also ist jedes Steiner-System ein Blockplan, und jeder Blockplan ist eine *taktische Konfiguration* (Moore [7], Carmichael [4, chapter 14]), d.h. eine endliche Inzidenzstruktur mit (P_1) und (B_1). Zu jeder solchen taktischen Konfiguration $\mathfrak{K}$ existiert ein durch sie eindeutig bestimmtes Paar (m, n) natürlicher Zahlen (ihr „Typ") mit folgenden Eigenschaften: $\mathfrak{K}$ erfüllt $(B_1), \ldots, (B_m)$ und $(P_1), \ldots, (P_n)$, aber kein (P_i) mit $n < i \leqq p_0$ und kein (B_j) mit $m < j \leqq b_0$. Es

liegt nun nahe zu fragen, welche Zahlenpaare wirklich als Typen von taktischen Konfigurationen auftreten können. Wir zeigen, daß [abgesehen vom Typ (b_0, p_0) — in diesem später immer ausgeschlossenen Fall inzidiert jeder Punkt mit jedem Block] genau die Typen $(1, n)$, $(n, 1)$ und (n, n) tatsächlich vorkommen (Satz 4). Für den Fall $n > 1$ ist Typ $(1, n)$ mit $b_0 > p_0$ äquivalent, dual dazu Typ $(n, 1)$ mit $p_0 > b_0$, und folglich Typ (n, n) mit $b_0 = p_0$ (Folgerung von Satz 3, man vergleiche hierzu FISHER [5], BOSE [2], RYSER [9]). Diese Klassifikation der taktischen Konfigurationen nach ihren Typen ist natürlich recht grob, z. B. hat die zur Zeit völlig unübersehbare Fülle der symmetrischen Blockpläne (PICKERT [8, S. 288]) mit $b_1 - b_2 > 1$ (dazu gehören u. a. sämtliche endlichen projektiven Ebenen) den gemeinsamen Typ $(2, 2)$. Um so überraschender ist die Tatsache, daß diese Fülle schon für $n = 3$ völlig verschwindet: Alle (nichttrivialen, s. o.) taktischen Konfigurationen vom Typ (n, n) mit $n > 2$ sind isomorph zu der mit $n + 1$ Punkten und den sämtlichen n-elementigen Punktmengen als Blöcken (Satz 5). Die ungleich schwierigere Frage, ob ein entsprechender Satz auch im unsymmetrischen Fall gilt, d. h. ob es eine natürliche Zahl k derart gibt, daß jede taktische Konfiguration vom Typ $(1, n)$ mit $n > k$ die Eigenschaft $b_1 = n$ hat, bleibt offen. Nach WITT [11] müßte ein solches k jedenfalls $\geqq 5$ sein.

Die Beweise aller Sätze dieser Arbeit beruhen auf dem alten Gedanken, gewisse Mengen von inzidenten Paaren auf zwei verschiedene Weisen abzuzählen. Bei dieser Idee ist die Freiheit, die man in der Auswahl der abzuzählenden Mengen hat, anscheinend bisher nicht immer genügend beachtet worden. Hier ergeben sich aus dem für die ganze Arbeit fundamentalen Lemma 1 (S. 260) unter anderem auch einige überraschend einfache Herleitungen bekannter Resultate; es sei hier nur auf den kurzen Beweis der Folgerung aus dem Hilfssatz von S. 269 hingewiesen, ein Ergebnis, das auch von MAJUMDAR durch längere Operationen mit Inzidenzmatrizen gewonnen worden ist [6, theorem 1]. Die Inzidenzmatrizen werden hier nirgends herangezogen.

Außer den erwähnten Sätzen über taktische Konfigurationen hat das grundlegende Lemma 1 noch viele weitere Konsequenzen, von denen hier noch die Ungleichungen (3.6) für die Invarianten b_t, p_j einer beliebigen endlichen Inzidenzstruktur hervorgehoben seien [diese Invarianten werden in 2 ohne die Axiome (P_i), (B_j) so als Mittelwerte erklärt, daß sie im Falle der Gültigkeit dieser Axiome die oben erklärte Bedeutung erhalten]. Daß diese Ungleichungen genau bei Gültigkeit von (B_1) und (P_1) alle zu Gleichungen werden (Satz 1), macht die Sonderstellung dieser Axiome unter den übrigen (P_i), (B_j) deutlich, die sich auch aus den an die Sätze 2 und 3 anschließenden Bemerkungen ergibt und die der späteren Beschränkung der Untersuchung auf taktische Konfigurationen Natürlichkeit verleiht.

2. Endliche Inzidenzstrukturen. Eine *Inzidenzstruktur* ist ein Tripel $\mathfrak{I} = (\mathfrak{P}, \mathfrak{B}, I)$ von Mengen mit den Eigenschaften

$$\mathfrak{P} \cap \mathfrak{B} = \emptyset \quad \text{und} \quad I \subseteq \mathfrak{P} \times \mathfrak{B}.$$

Die Elemente von $\mathscr{P}$ heißen *Punkte*, die von $\mathscr{B}$ *Blöcke*, und die von I *inzidente Paare*. In dieser Arbeit werden nur *endliche* Inzidenzstrukturen betrachtet, d.h. solche, bei denen sowohl $\mathscr{P}$ als auch $\mathscr{B}$, und deshalb auch I, endliche Mengen sind. Wir verabreden für alles weitere folgende Bezeichnungen: Punkte werden durch kleine, Blöcke durch große lateinische Buchstaben bezeichnet, Mengen von Punkten bzw. Mengen von Blöcken im allgemeinen durch kleine bzw. große deutsche Buchstaben; nur für die Mengen *aller* Punkte bzw. *aller* Blöcke behalten wir die Bezeichnungen $\mathscr{P}$ und $\mathscr{B}$ bei. Striche bezeichnen Komplementärmengen:

$$\mathfrak{p}' = \mathscr{P} - \mathfrak{p}, \qquad \mathfrak{B}' = \mathscr{B} - \mathfrak{B}.$$

Statt $(p, B) \in I$ schreiben wir auch $p\,I\,B$ oder $B\,I\,p$, und für diesen Sachverhalt werden die üblichen Sprechweisen wie „p und B sind inzident", „p liegt auf B", „B geht durch p" benutzt. Weiter wird $\mathfrak{p}\,I\,\mathfrak{B}$ oder $\mathfrak{B}\,I\,\mathfrak{p}$ statt $\mathfrak{p} \times \mathfrak{B} \subseteq I$ geschrieben. Absolutstriche bezeichnen wie üblich Mächtigkeiten von Mengen.

Ist $\mathfrak{B}$ eine beliebige Block- und $\mathfrak{p}$ eine beliebige Punktmenge, so bedeutet $[\mathfrak{B}, \mathfrak{p}]$ die Anzahl aller $B \in \mathfrak{B}$ mit $B\,I\,\mathfrak{p}$, und $[\mathfrak{p}, \mathfrak{B}]$ ist dual erklärt. Besonders häufig kommen die Zahlen $[\mathscr{B}, \mathfrak{p}]$ und $[\mathscr{P}, \mathfrak{B}]$ vor; dafür schreiben wir kürzer $[\mathfrak{p}]$ bzw. $[\mathfrak{B}]$, also:

$[\mathfrak{p}]$ = *Anzahl der mit jedem* $p \in \mathfrak{p}$ *inzidenten Blöcke.* | $[\mathfrak{B}]$ = *Anzahl der mit jedem* $B \in \mathfrak{B}$ *inzidenten Punkte.*

Diese Bezeichnungsweise ist insofern nicht ganz präzise, als das Symbol $[\emptyset]$ sowohl die Anzahl aller Punkte als auch die aller Blöcke bedeuten kann, je nachdem ob $[\mathscr{P}, \emptyset]$ oder $[\mathscr{B}, \emptyset]$ gemeint ist. Hieraus werden aber im späteren Text nirgends Unklarheiten entstehen.

Aus diesen Festsetzungen ergibt sich unmittelbar die Richtigkeit der folgenden Beziehungen:

$$(2.1) \qquad \begin{cases} [\mathfrak{B}, \mathfrak{p}] + [\mathfrak{B}', \mathfrak{p}] = [\mathfrak{p}], \qquad [\mathfrak{p}, \mathfrak{B}] + [\mathfrak{p}', \mathfrak{B}] = [\mathfrak{B}], \\ [\mathfrak{B}, \emptyset] = |\mathfrak{B}|, \quad [\mathfrak{p}, \emptyset] = |\mathfrak{p}|, \qquad [\emptyset, \mathfrak{B}] = [\emptyset, \mathfrak{p}] = 0, \end{cases}$$

für beliebige $\mathfrak{B} \subseteq \mathscr{B}$ und $\mathfrak{p} \subseteq \mathscr{P}$.

Zu jeder endlichen Inzidenzstruktur $\mathscr{I} = (\mathscr{P}, \mathscr{B}, I)$ definieren wir die folgenden *Invarianten* b_i *und* p_j $(i = 0, \ldots, |\mathscr{P}|,\ j = 0, \ldots, |\mathscr{B}|)$: b_i ist der Mittelwert der Zahlen $[\mathfrak{x}]$, wobei $\mathfrak{x}$ alle Punktmengen mit $|\mathfrak{x}| = i$ durchläuft, und p_j ist dual erklärt. Insbesondere ist $b_0 = |\mathscr{B}|$ die Anzahl aller Blöcke und $p_0 = |\mathscr{P}|$ die Anzahl aller Punkte, und

$$(2.2) \quad b_i = \binom{p_0}{i}^{-1} \sum_{|\mathfrak{x}|=i} [\mathfrak{x}], \quad i = 1, \ldots, p_0; \qquad p_j = \binom{b_0}{j}^{-1} \sum_{|\mathfrak{X}|=j} [\mathfrak{X}], \quad j = 1, \ldots, b_0.$$

Die zwischen den b_i und p_j bestehenden Beziehungen werden im nächsten Paragraphen behandelt.

Mit Hilfe der Zahlen b_i und p_j lassen sich die in der Einleitung ausgesprochenen Axiome (P_i) und (B_j) jetzt kürzer und präziser folgendermaßen formulieren:

(P_i) *Es ist* $b_i > 0$, *und für jedes* $\mathfrak{x}$ *mit* $|\mathfrak{x}| = i$ *gilt* $[\mathfrak{x}] = b_i$ $(i = 1, \ldots, p_0)$.

(B_j) *Es ist* $p_j > 0$, *und für jedes* $\mathfrak{X}$ *mit* $|\mathfrak{X}| = j$ *gilt* $[\mathfrak{X}] = p_j$ $(j = 1, \ldots, b_0)$.

Bei Gültigkeit von (P_i) bzw. (B_i) haben die Zahlen $[\mathfrak{x}]$ mit $|\mathfrak{x}| = i$ bzw. die Zahlen $[\mathfrak{X}]$ mit $|\mathfrak{X}| = j$ also die Streuung Null um ihren von Null verschiedenen Mittelwert. (P_0) und (B_0) bedeuten nur, daß die Mengen $\mathscr{B}$ bzw. $\mathscr{P}$ nicht leer sind; diese beinahe selbstverständliche Voraussetzung machen wir von jetzt an immer, ohne sie besonders zu erwähnen. Die möglichen Abhängigkeiten zwischen den (P_i) und den (B_j) werden in den Paragraphen **4** und **5** erörtert werden.

Für eine spätere Anwendung brauchen wir noch den folgenden Begriff. Jede endliche Inzidenzstruktur $\mathscr{I} = (\mathscr{P}, \mathscr{B}, I)$ bestimmt in natürlicher Weise die folgenden *abgeleiteten Strukturen* $\mathscr{I}_{ij} = (\mathscr{P}_i, \mathscr{B}_j, I_{ij})$, $i = 1, \ldots, p_0$, $j = 1, \ldots, b_0$: $\mathscr{P}_i$ ist die Menge der i-elementigen Teilmengen von $\mathscr{P}$, weiter $\mathscr{B}_j$ die der j-elementigen Teilmengen von $\mathscr{B}$; und $\mathfrak{p} I_{ij} \mathfrak{B}$, wo $\mathfrak{p} \in \mathscr{P}_i$ und $\mathfrak{B} \in \mathscr{B}_j$, bedeutet dasselbe wie $\mathfrak{p} I \mathfrak{B}$ (nämlich $\mathfrak{p} \times \mathfrak{B} \subseteq I$, vgl. S. 258). Natürlich ist $\mathscr{I}_{11}$ dasselbe wie die Ausgangsstruktur $\mathscr{I}$. Für jedes $\mathfrak{p} \in \mathscr{P}_i$ wird die Anzahl der $\mathfrak{B} \in \mathscr{B}_j$ mit $\mathfrak{p} I_{ij} \mathfrak{B}$ durch $[\mathfrak{p}]_j$ bezeichnet, da wir die Bezeichnung $[\mathfrak{p}]$ für die Anzahl der $B \in \mathscr{B}$ mit $B I \mathfrak{p}$ beibehalten wollen, und $[\mathfrak{B}]_i$ wurd dual definiert. Natürlich ist $[\mathfrak{p}]_1 = [\mathfrak{p}]$ und $[\mathfrak{B}]_1 = [\mathfrak{B}]$, und allgemeiner gilt

$$[\mathfrak{p}]_j = \binom{[\mathfrak{p}]}{j}, \qquad [\mathfrak{B}]_i = \binom{[\mathfrak{B}]}{i}, \tag{2.3}$$

wie aus der Definition von $\mathscr{I}_{ij}$ sofort folgt.

3. Die Invarianten-Ungleichungen. In einer beliebigen endlichen Inzidenzstruktur seien eine Blockmenge $\mathfrak{B}$ und eine Punktmenge $\mathfrak{p}$ gegeben, die auch leer sein dürfen. Dann läßt sich die Anzahl der inzidenten Paare (x, X) mit $x \mathrel{I} \mathfrak{B}$ und $X \mathrel{I} \mathfrak{p}$ auf zwei verschiedene Weisen abzählen: Auf einem Block X, der mit jedem Punkt von $\mathfrak{p}$ inzidiert, liegen genau $[\mathfrak{B} \cup \{X\}]$ verschiedene mit jedem Block von $\mathfrak{B}$ inzidente Punkte. Also ist die fragliche Anzahl gleich

$$\sum_{X I \mathfrak{p}} [\mathfrak{B} \cup \{X\}].$$

Von den $[\mathfrak{p}]$ verschiedenen Summanden dieser Summe haben $[\mathfrak{B}, \mathfrak{p}]$, nämlich die mit $X \in \mathfrak{B}$, den Wert $[\mathfrak{B}]$, und deshalb ist die Summe gleich

$$[\mathfrak{B}]\,[\mathfrak{B}, \mathfrak{p}] + \sum_{\mathfrak{p} I X \in \mathfrak{B}'} [\mathfrak{B} \cup \{X\}].$$

Die duale Abzählung liefert ein analoges Resultat, und damit ist der folgende wichtige Hilfssatz bewiesen:

Lemma 1. *Für jede Blockmenge $\mathfrak{B}$ und jede Punktmenge $\mathfrak{p}$ einer beliebigen endlichen Inzidenzstruktur gilt:*

$$(3.1) \qquad [\mathfrak{B}]\,[\mathfrak{B},\mathfrak{p}]+\sum_{\mathfrak{p}\,I\,X\in\mathfrak{B}'}[\mathfrak{B}\cup\{X\}]=[\mathfrak{p}]\,[\mathfrak{p},\mathfrak{B}]+\sum_{\mathfrak{B}\,I\,x\in\mathfrak{p}'}[\mathfrak{p}\cup\{x\}].$$

Wie üblich sind dabei leere Summen gleich Null zu setzen.

Die fundamentale Gleichung (3.1) wird später — für verschiedene spezielle Auswahlen von $\mathfrak{B}$ und $\mathfrak{p}$ — mehrfach angewandt werden. Wir geben gleich einige Folgerungen. Für $\mathfrak{B}=\mathfrak{p}=\emptyset$ geht (3.1) in $\sum[X]=\sum[x]$ über, wobei über alle Blöcke X und alle Punkte x zu summieren ist. Faßt man diese Gleichung als eine Beziehung für die abgeleitete Struktur $\mathcal{I}_{ij}$ anstatt für $\mathcal{I}$ auf, so hat man $[X]$ nach (2.3) durch $\binom{[\mathfrak{X}]}{i}$ und $[x]$ durch $\binom{[\mathfrak{x}]}{j}$ zu ersetzen, mit $|\mathfrak{x}|=i$ und $|\mathfrak{X}|=j$. Man erhält also die allgemeinere Beziehung

$$(3.2) \qquad \sum_{|\mathfrak{X}|=j}\binom{[\mathfrak{X}]}{i}=\sum_{|\mathfrak{x}|=i}\binom{[\mathfrak{x}]}{j} \quad i=0,\ldots,p_0,\quad j=0,\ldots,b_0.$$

(Daß das auch für $i=0$ und $j=0$ richtig ist, folgt noch nicht aus dem vorhergehenden, läßt sich aber leicht direkt verifizieren.) Für $j=1$ bzw. $i=1$ erhält man aus (3.2) unter Beachtung der Definitionen (2.2) noch die für später wichtigen Gleichungen:

$$(3.3) \qquad \sum_{X\in\mathfrak{B}}\binom{[X]}{i}=\binom{p_0}{i}b_i\,,\quad i=0,\ldots,p_0,\qquad \sum_{x\in\mathfrak{P}}\binom{[x]}{j}=\binom{b_0}{j}p_j,\quad j=0,\ldots,b_0.$$

Ohne Zuhilfenahme der abgeleiteten Strukturen lassen sich (3.2) oder (3.3) nur mühsam aus Lemma 1 herleiten.

Als weitere Folgerung aus (3.1) beweisen wir noch die beiden nachstehenden Gleichungen, die wir später mit (3.3) kombinieren werden:

$$(3.4) \qquad \begin{cases} \displaystyle\sum_{X\in\mathfrak{B}}\binom{[X]}{i}[X]=\binom{p_0}{i}[(p_0-i)\,b_{i+1}+i\,b_i], & i=0,\ldots,p_0-1,\\ \displaystyle\sum_{x\in\mathfrak{P}}\binom{[x]}{j}[x]=\binom{b_0}{j}[(b_0-j)\,p_{j+1}+j\,p_j], & j=0,\ldots,b_0-1. \end{cases}$$

Beweis. Aus Dualitätsgründen genügt es, die Richtigkeit der ersten Gleichung (3.4) zu zeigen. Sei $\mathfrak{x}$ eine beliebige Punktmenge mit $0\leqq|\mathfrak{x}|=i<p_0$. Dann liefert (3.1) mit $\mathfrak{B}=\emptyset$ und $\mathfrak{p}=\mathfrak{x}$:

$$\sum_{X\,I\,\mathfrak{x}}[X]=i\,[\mathfrak{x}]+\sum_{x\in\mathfrak{x}'}[\mathfrak{x}\cup\{x\}].$$

Die $\binom{p_0}{i}$ verschiedenen solchen Gleichungen, die man für alle $\mathfrak{x}$ mit $|\mathfrak{x}|=i$ erhält, werden nun addiert. Da zunächst

$$\sum_{|\mathfrak{x}|=i}\Big(\sum_{X\,I\,\mathfrak{x}}[X]\Big)=\sum_{X\in\mathfrak{B}}\binom{[X]}{i}[X]$$

ist (jedes $[X]$ muß ebenso oft gezählt werden, wie es verschiedene $\mathfrak{x}$ mit $|\mathfrak{x}| = i$ und $X I \mathfrak{x}$ gibt), da weiter nach (2.2)

$$\sum_{|\mathfrak{x}|=i} [\mathfrak{x}] = b_i \binom{p_0}{i}$$

ist, und da schließlich, wieder wegen (2.2),

$$\sum_{|\mathfrak{x}|=i} \Big(\sum_{x \in \mathfrak{x}'} [\mathfrak{x} \cup \{x\}] \Big) = (i+1) \sum_{|\mathfrak{y}|=i+1} [\mathfrak{y}] = (i+1)\, b_{i+1} \binom{p_0}{i+1}$$

gilt (jedes $[\mathfrak{y}]$ mit $|\mathfrak{y}| = i+1$ muß ebenso oft gezählt werden, wie $\mathfrak{y}$ auf verschiedene Weisen in der Form $\mathfrak{x} \cup \{x\}$ darstellbar ist, also für jedes der $i+1$ verschiedenen $x \in \mathfrak{y}$ je einmal), ergibt die Summation:

$$\sum_{X \in \mathfrak{B}} \binom{[X]}{i} [X] = i\, b_i \binom{p_0}{i} + (i+1)\, b_{i+1} \binom{p_0}{i+1}.$$

Hieraus ergibt sich nun (3.4) durch triviale Umformungen der rechten Seite.

Wir kommen jetzt zum eigentlichen Thema dieses Paragraphen, nämlich zu den zwischen den Invarianten b_i und p_j einer endlichen Inzidenzstruktur bestehenden Beziehungen. Wir zeigen zuerst: *Es gilt*

(3.5) $b_i \geqq b_{i+1}$ *für* $i = 0, \ldots, p_0 - 1$ *und* $p_j \geqq p_{j+1}$ *für* $j = 0, \ldots, b_0 - 1$,

und Gleichheit herrscht genau für diejenigen i *bzw.* j, *für welche es keinen Block* B *mit* $i \leqq [B] < p_0$ *bzw. keinen Punkt* p *mit* $j \leqq [p] < b_0$ *gibt.*

Beweis. Sei $0 \leqq i < p_0$ und $b_i \leqq b_{i+1}$. Dann folgt aus (3.4) und (3.3):

$$\sum_{X \in \mathfrak{B}} \binom{[X]}{i} [X] = \binom{p_0}{i} [(p_0 - i)\, b_{i+1} + i\, b_i] \geqq \binom{p_0}{i} p_0 b_i = p_0 \sum_{X \in \mathfrak{B}} \binom{[X]}{i}.$$

Aber $p_0 \geqq [X]$ für alle $X \in \mathfrak{B}$, also kann nur Gleichheit herrschen. Daraus folgt die Unmöglichkeit von $b_i < b_{i-1}$, so daß die Richtigkeit der Behauptung $b_i \geqq b_{i+1}$ bewiesen ist. Weiter ergibt sich, daß bei $b_i = b_{i+1}$ die Gleichung

$$\sum_{X \in \mathfrak{B}} (p_0 - [X]) \binom{[X]}{i} = 0$$

gelten muß; umgekehrt hat diese Gleichung, wieder wegen (3.3) und (3.4), die Gleichheit von b_i und b_{i+1} zur Folge. Da die Glieder dieser Summe alle nichtnegativ sind, müssen sie einzeln verschwinden; es gilt also: Genau dann ist $b_i = b_{i+1}$, wenn $p_0 = [X]$ oder $\binom{[X]}{i} = 0$ für jeden Block X gilt. Aber die zweite Alternative ist mit $[X] < i$ äquivalent. Damit (und durch duale Schlüsse) sind alle Behauptungen über (3.5) bewiesen.

Eine weitere, und für das spätere besonders wichtige, Kette von Ungleichungen liefert der folgende Satz:

Satz 1. *Die zu einer beliebigen endlichen Inzidenzstruktur gehörigen Invarianten b_i und p_j genügen den folgenden Ungleichungen:*

$$(3.6)\quad \begin{cases} b_i(p_1 - i) \leqq b_{i+1}(p_0 - i), \\ i = 0, \ldots, p_0 - 1. \end{cases} \qquad \begin{cases} p_j(b_1 - j) \leqq p_{j+1}(b_0 - j), \\ j = 0, \ldots, b_0 - 1. \end{cases}$$

Ist das Axiom (B_1) erfüllt, so gilt für alle i sogar Gleichheit. Ist (B_1) nicht erfüllt, so gilt Gleichheit genau für $i=0$ und $i>\max([X], X\in\mathscr{B})$.

Ist das Axiom (P_1) erfüllt, so gilt für alle j sogar Gleichheit. Ist (P_1) nicht erfüllt, so gilt Gleichheit genau für $j=0$ und $j>\max([x], x\in\mathscr{P})$.

Beweis. Aus Dualitätsgründen genügt es wieder, die Behauptungen über die erste der Ungleichungen (3.6) zu beweisen. Aus den Folgerungen (3.3) und (3.4) von Lemma 1 ergibt sich unmittelbar

$$\sum_{X\in\mathscr{B}} \binom{[X]}{i} ([X] - p_1) = \binom{p_0}{i} [(p_0 - i)\, b_{i+1} - (p_1 - i)\, b_i]$$

für jedes in Frage kommende i. Nun ist p_1 der Mittelwert der nicht negativen Zahlen $[X]$, und $\binom{[X]}{i}$ ist eine für alle $[X]$ erklärte und dort monoton steigende Funktion von $[X]$. Da $\binom{p_0}{i}$ positiv ist, ergibt sich die Behauptung also aus dem folgenden Hilfssatz, den wir der Vollständigkeit halber hier beweisen wollen, obwohl er gar nicht tief liegt und auch mit endlichen Inzidenzstrukturen nicht viel zu tun hat:

Es seien $x_1, \ldots, x_n$ nichtnegative reelle Zahlen und f eine auf den x_i erklärte, dort nichtnegative und (schwach) monoton steigende reellwertige Funktion. Dann gilt:

$$(3.7)\qquad \sum_{i=1}^{n} f(x_i)\,(x_i - \bar{x}) \geqq 0,$$

wobei $\bar{x}$ den Mittelwert $n^{-1}\sum_{i=1}^{n} x_i$ der x_i bedeutet; Gleichheit für (3.7) tritt genau in den folgenden Fällen ein:

(i) $x_i = \bar{x}$ *für alle i,* (ii) $f(x_1) = \cdots = f(x_n)$.

Beweis. Daß (3.7) in den Fällen (i) und (ii) mit dem Gleichheitszeichen gilt, ist trivial. Also bleibt zu zeigen: Gilt weder (i) noch (ii), so ist der Ausdruck (3.7) positiv. Ohne Beschränkung der Allgemeinheit können die x_i als der Größe nach geordnet angenommen werden:

$$x_1 \geqq \cdots \geqq x_n.$$

Da (i) nicht gilt, ist $n>1$, und es gibt ein wohlbestimmtes r mit $1\leqq r<n$ derart, daß $x_i \geqq \bar{x}$ für $i=1, \ldots, r$ und $x_i<\bar{x}$ für $i=r+1, \ldots, n$ gilt. Die Summe in (3.7) ist dann gleich

$$(a)\qquad \sum_{i=1}^{r} f(x_i)\,(x_i - \bar{x}) - \sum_{i=r+1}^{n} f(x_i)\,(\bar{x} - x_i),$$

und diese beiden Summen haben nur nichtnegative Summanden. Wegen der Monotonie von f gilt $f(x_i) \geqq f(x_r)$ für $i = 1, \ldots, r$ und $f(x_{r+1}) \geqq f(x_i)$ für $i = r+1, \ldots, n$, und folglich ist der Ausdruck (a) jedenfalls nicht kleiner als

$$\text{(b)} \qquad f(x_r) \sum_{i=1}^{r} (x_i - \bar{x}) - f(x_{r+1}) \sum_{i=r+1}^{n} (\bar{x} - x_i).$$

Ist nun $f(x_r) = f(x_{r-1})$, so gilt sogar (a) $>$ (b), da sonst der Fall (ii) vorläge. In diesem Falle sind wir also fertig. Ist aber $f(x_r) > f(x_{r+1})$, so ist der Ausdruck (b) größer als

$$f(x_{r-1}) \sum_{i=1}^{n} (x_i - \bar{x}) = 0,$$

und folglich ist die Behauptung auch dann richtig. Damit ist der Hilfssatz bewiesen. Also gelten die Ungleichungen (3.6) immer. Das Axiom (B_1) bedeutet gerade das Eintreten des Falles (i) des Hilfssatzes, und genau für $i = 0$ oder $i > \max([X], X \in \mathfrak{B})$ sind alle $\binom{[X]}{i}$ gleich (nämlich $= 1$ bzw. $= 0$), d.h. genau für diese i tritt Fall (ii) ein. Damit ist Satz 1 bewiesen.

Die Beziehungen (3.6) werden später vor allem für $i, j = 0, 1$ und bei Gültigkeit von (B_1) und (P_1) gebraucht. Wir notieren diese speziellen Gleichungen noch einmal gesondert: *Bei Gültigkeit von* (B_1) *und* (P_1) *ist*

$$\text{(3.8)} \qquad b_0 p_1 = b_1 p_0, \quad b_1(p_1 - 1) = b_2(p_0 - 1), \quad p_1(b_1 - 1) = p_2(b_0 - 1).$$

4. Die Axiome (P_i) und (B_j). Wir kommen jetzt zu der Frage nach den zwischen den Axiomen (P_i) und (B_j) bestehenden Abhängigkeiten. Für den Fall der Gültigkeit von (P_1) oder (B_1) gibt der folgende (im wesentlichen auch von SPROTT [*10*, S. 753] bewiesene) Satz eine befriedigende Antwort:

Satz 2. *Aus* (B_1) *und* (P_n), $0 < n \leqq p_0$, *folgen* $(P_1), \ldots, (P_{n-1})$. | *Aus* (P_1) *und* (B_m), $0 < m \leqq b_0$, *folgen* $(B_1), \ldots, (B_{m-1})$.

Es genügt wieder, die erste Behauptung zu beweisen, und dazu ist es nur notwendig zu zeigen, daß (P_k) aus (B_1) und (P_{k+1}) folgt $(k = n-1, \ldots, 1)$: Wegen $k+1 \leqq n \leqq p_0$ existiert sicher eine Punktmenge $\mathfrak{p}$ mit $|\mathfrak{p}| = k$. Für dieses $\mathfrak{p}$ und die leere Blockmenge $\mathfrak{B} = \emptyset$ liefert die Gleichung (3.1) von Lemma 1:

$$[\mathfrak{p}](p_1 - k) = b_{k+1}(p_0 - k),$$

denn wegen (B_1) ist $[X] = p_1$ für jedes $X \in \mathfrak{B}$, und wegen (P_{k+1}) ist $[\mathfrak{p} \cup \{x\}] = b_{k+1}$ für jedes $x \in \mathfrak{p}'$. Da wegen (P_{k+1}) sicher $p_1 > k$ ist, ist $[\mathfrak{p}]$ durch diese Gleichung eindeutig bestimmt, größer als Null und nur von $|\mathfrak{p}| = k$, nicht von $\mathfrak{p}$ abhängig. Also gilt (P_k) mit $b_k = [\mathfrak{p}]$, entsprechend Satz 1. Damit ist Satz 2 bewiesen.

Die ausgearteten endlichen projektiven Ebenen zeigen, daß die Voraussetzung der Gültigkeit von (B_1) nicht weggelassen oder etwa durch die eines (B_m) mit $m < n$ oder ähnliches ersetzt werden darf. Daß man in vielen

Fällen aber auch ohne die Voraussetzung von (B_1) oder (P_1) aus gewissen (P_i), (B_j) andere folgern kann, zeigen die folgenden Überlegungen.

Es sei $\mathfrak{B}$ eine Blockmenge mit $|\mathfrak{B}| = m-1 \geqq 0$. Dann gilt

$$\sum_{\substack{\mathfrak{x} I \mathfrak{B} \\ |\mathfrak{x}|=n}} \Big(\sum_{\mathfrak{x} I X \in \mathfrak{B}'} [\mathfrak{B} \cup \{X\}] \Big) = \sum_{X \in \mathfrak{B}'} \binom{[\mathfrak{B} \cup \{X\}]}{n} [\mathfrak{B} \cup \{X\}], \tag{4.1}$$

denn für jedes $X \in \mathfrak{B}'$ muß $[\mathfrak{B} \cup \{X\}]$ ebenso oft gezählt werden, wie es verschiedene $\mathfrak{x}$ mit $|\mathfrak{x}| = n$ und $\mathfrak{x} I \mathfrak{B}$ gibt, die außerdem $\mathfrak{x} I X$ erfüllen, also genau $\binom{[\mathfrak{B} \cup \{X\}]}{n}$ mal. Ist nun (B_m) erfüllt, so nimmt (4.1) die einfachere Gestalt

$$\sum_{\substack{\mathfrak{x} I \mathfrak{B} \\ |\mathfrak{x}|=n}} \big([\mathfrak{x}] - m + 1 \big) = \binom{p_m}{n} (b_0 - m + 1) \tag{4.2}$$

an, wie aus den Beziehungen (2.1) wegen der vorausgesetzten Eigenschaften der $\mathfrak{x}$ folgt. Aus (4.2) sowie aus der dazu dualen Beziehung ergibt sich nun unmittelbar der folgende Satz:

Erfüllt eine endliche Inzidenzstruktur die Axiome (B_m) und (P_n), so gilt

$$\begin{cases} \binom{[\mathfrak{B}]}{n} (b_n - m + 1) = \binom{p_m}{n} (b_0 - m + 1) \\ \textit{und} \\ \binom{[\mathfrak{p}]}{m} (p_m - n + 1) = \binom{b_n}{m} (p_0 - n + 1) \end{cases} \tag{4.3}$$

für jede Blockmenge $\mathfrak{B}$ mit $|\mathfrak{B}| = m - 1$ und jede Punktmenge $\mathfrak{p}$ mit $|\mathfrak{p}| = n - 1$ $(m, n > 0)$.

Diese Beziehungen zeigen, daß die Zahlen $[\mathfrak{B}]$ bzw. $[\mathfrak{p}]$ mit $|\mathfrak{B}| = m-1$ und $|\mathfrak{p}| = n-1$ bei Gültigkeit von (B_m) und (P_n) gewissen Gleichungen n-ten bzw. m-ten Grades genügen müssen, also höchstens n bzw. m verschiedener Werte fähig sind [vorausgesetzt allerdings, daß die Koeffizienten $b_n - m + 1$ bzw. $p_m - n + 1$ nicht verschwinden; aus diesem Grund kann man z.B. Satz 2 nicht ohne weiteres aus (4.3) herleiten]. Da diese Zahlen natürlich außerdem positiv und ganz sein müssen, kann man in manchen Fällen zeigen, daß statt n bzw. m tatsächlich nur ein Wert für $[\mathfrak{B}]$ bzw. $[\mathfrak{p}]$ möglich ist, so daß die Gültigkeit von (B_{m-1}) bzw. (P_{n-1}) folgt. Wir zeigen dies noch für die Fälle $m = 2$ und $n = 2$.

Aus (B_2) und (P_n) mit $n < p_2 + 1$ und $b_n > 1$ folgen (B_1) und $(P_1), \ldots, (P_{n-1})$.

Aus (P_2) und (B_m) mit $m < b_2 + 1$ und $p_m > 1$ folgen (P_1) und $(B_1), \ldots, (B_{m-1})$.

Wir brauchen wieder nur die erste Behauptung zu beweisen. Aus der zweiten Gleichung (4.3) mit $m = 2$ folgt, daß alle die Zahlen $[\mathfrak{p}]$ für $|\mathfrak{p}| = n - 1$ der quadratischen Gleichung

$$(p_2 - n + 1)(x^2 - x) = b_n (b_n - 1)(p_0 - n + 1) \tag{4.4}$$

genügen. Diese Gleichung hat die Wurzeln $\frac{1}{2}(1 \pm \sqrt{1+4K})$, mit

$$K = b_n(b_n - 1)(p_0 - n + 1)(p_2 - n + 1)^{-1}.$$

Aus den Voraussetzungen über p_2 und b_n folgt, daß K wohldefiniert und positiv ist. Eine der Wurzeln von (4.4) fällt also immer negativ aus, und es folgt

$$[\mathfrak{p}] = \tfrac{1}{2}(1 + \sqrt{1+4K}) > 0$$

unabhängig von $\mathfrak{p}$, und daher gilt (P_{n-1}). Da mit $p_2 - n + 1$ natürlich auch alle $p_2 - j + 1$ positiv sind und da weiter aus $b_n > 1$ wegen (3.5) auch $b_j > 1$ folgt $(j = 0, \ldots, n-1)$, kann man dieselbe Schlußweise fortgesetzt weiter anwenden und erhält nacheinander $(P_{n-2}), \ldots, (P_1)$. Aus Satz 2 ergibt sich dann noch die Gültigkeit von (B_1), und damit ist alles bewiesen.

Man kann die Beziehung (4.1), kombiniert mit Lemma 1, noch zum Beweis eines weiteren Satzes benutzen, der eine teilweise Antwort auf die Frage liefert, wie die Axiome (P_i) und (B_j) das Verhältnis der Punkteanzahl p_0 zur Blöckeanzahl b_0 beeinflussen. Der in (4.1) auftretende Ausdruck $\sum_{\mathfrak{x} I X \in \mathfrak{B}'} [\mathfrak{B} \cup \{X\}]$ mit $\mathfrak{x} I \mathfrak{B}$ und $|\mathfrak{x}| = n$ ist nach Lemma 1 und (2.1) gleich

$$n[\mathfrak{x}] - |\mathfrak{B}|[\mathfrak{B}] + \sum_{\mathfrak{B} I x \in \mathfrak{x}'} [\mathfrak{x} \cup \{x\}]$$

(hier ist wieder $|\mathfrak{B}|$ statt $m-1$ geschrieben), so daß (4.1) jetzt

$$n \sum_{\substack{\mathfrak{x} I \mathfrak{B} \\ |\mathfrak{x}| = n}} [\mathfrak{x}] - |\mathfrak{B}|[\mathfrak{B}] \binom{[\mathfrak{B}]}{n} + \sum_{\substack{\mathfrak{x} I \mathfrak{B} \\ |\mathfrak{x}| = n}} \Big(\sum_{\mathfrak{B} I x \in \mathfrak{x}'} [\mathfrak{x} \cup \{x\}] \Big) = \sum_{X \in \mathfrak{B}'} \binom{[\mathfrak{B} \cup \{X\}]}{n} [\mathfrak{B} \cup \{X\}]$$

liefert. Diese komplizierte Beziehung geht *bei Gültigkeit von* (P_n) *und* (P_{n+1}) *in*

$$\sum_{X \in \mathfrak{B}'} \binom{[\mathfrak{B} \cup \{X\}]}{n} [\mathfrak{B} \cup \{X\}] = \binom{[\mathfrak{B}]}{n} [n(b_n - b_{n+1}) + [\mathfrak{B}](b_{n+1} - |\mathfrak{B}|)] \tag{4.5}$$

über, und damit läßt sich der erwähnte Satz beweisen. Er lautet:

Satz 3. *Gegeben sei eine endliche Inzidenzstruktur mit* $1 < p_0 > p_1 > 0$ *und* $1 < b_0 > b_1 > 0$.

Gilt (B_1) *und ein* (P_n) *mit* $n > 1$, *so ist* $b_0 \geqq p_0$. *Gleichheit herrscht genau dann, wenn ein* (B_m) *mit* $m > 1$ *erfüllt ist.*

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Es genügt, die erste Behauptung zu beweisen. Andere Beweise dafür, oder für Teile davon, sind von FISHER [5], BOSE [2], RYSER [9] und MAJUMDAR [6] gegeben worden.

Es sei B ein zunächst beliebiger Block und $\mathfrak{B} = \{B\}$, also $|\mathfrak{B}| = 1$. Da nach Voraussetzung (B_1) gilt, ist dann $[\mathfrak{B}] = p_1$. Weiter folgt aus Satz 2 und der Voraussetzung $n > 1$ die Gültigkeit von (P_1) und (P_2), so daß (4.5) mit $n = 1$ anwendbar ist. Es ergibt sich

$$\sum_{X \neq B} [B, X]^2 = p_1[b_1 - b_2 + p_1(b_2 - 1)].$$

Außerdem liefert Lemma 1 mit $\mathfrak{p}=\emptyset$ wegen (P_1) sofort

$$\sum_{X \neq B} [B, X] = p_1 (b_1 - 1),$$

und daher gilt

$$(4.6) \quad \sum_{X \neq B} ([B, X] - p_2)^2 = p_1 [b_1 - b_2 + p_1 (b_2 - 1)] - 2 p_1 p_2 (b_1 - 1) + p_2^2 (b_0 - 1).$$

Die rechte Seite von (4.6) ist nun aber, wie man mit Hilfe der wegen (P_1) und (B_1) anwendbaren Gleichungen (3.8) nachrechnet, gleich dem Ausdruck

$$(4.7) \quad \frac{p_1 b_1 (p_0 - p_1)^2 (b_0 - p_0)}{p_0 (p_0 - 1)(b_0 - 1)};$$

die Einzelheiten dieser etwas langwierigen, aber nicht schwierigen Rechnung seien dem Leser überlassen. Also ist (4.7) gleich der Quadratsumme in (4.6) und daher nicht negativ. Daraus folgt aber sofort, daß $b_0 \geqq p_0$ sein muß.

Ist nun ein (B_m) mit $m>1$ erfüllt, so wegen Satz 2 auch (B_2), und dann sind alle $[B, X]=p_2$ (für $X \neq B$), so daß die Quadratsumme in (4.6) und damit auch (4.7) verschwindet. Dann folgt aber wegen der Voraussetzung $1<p_0>p_1>0$, daß $b_0=p_0$ sein muß. Gilt aber kein (B_m) mit $m>1$, so auch nicht (B_2), und es existieren vier Blöcke S, T, U, V mit $[S, T] \neq [U, V]$. Ist $[S, T] \neq [S, U]$, so setzen wir $B=S$; und ist $[S, T]=[S, U]$, so muß $[S, U] \neq [U, V]$ sein, und dann setzen wir $B=U$. Also läßt sich dann B jedenfalls so wählen, daß die $[B, X]$ mit $X \neq B$ nicht alle gleich sind. Dann ist die Quadratsumme in (4.6) und damit (4.7) aber positiv, und es folgt $b_0>p_0$. Damit ist alles bewiesen. Der Beweis liefert zusätzlich die folgende Aussage:

Gilt (B_1) und (P_n) mit $n>1$, und ist p_2 ganzzahlig, so ist $p_0(p_0-1)(b_0-1)$ ein Teiler von $p_1 b_1 (p_0-p_1)^2 (b_0-p_0)$.

Gilt (P_1) und (B_m) mit $m>1$, und ist b_2 ganzzahlig, so ist $b_0(b_0-1)(p_0-1)$ ein Teiler von $b_1 p_1 (b_0-b_1)^2 (p_0-b_0)$.

Denn dann ist die linke Seite von (4.6) und deshalb auch (4.7) eine ganze Zahl. Diese Aussage ist natürlich nur bei Nichterfülltsein von (B_2) bzw. (P_2) interessant, aber aus der Ganzzahligkeit von p_2 bzw. b_2 brauchen (B_2) bzw. (P_2) ja keineswegs zu folgen.

Daß die Voraussetzungen $n>1$ bzw. $m>1$ in Satz 3 unentbehrlich sind, zeigen viele bekannte Beispiele. Auch die Voraussetzung von (B_1) bzw. (P_1) ist notwendig: es gibt ausgeartete projektive Ebenen mit $p_0 \neq b_0$; bei diesen ist $[X]=1$ für alle Blöcke X, mit höchstens einer Ausnahme. In diesem Zusammenhang ist der folgende Satz von DE BRUIJN und ERDÖS [3] interessant: *Aus (P_2) mit $b_2=1$ und $[X]>1$ für jeden Block X folgt $b_0 \geqq p_0$, und Gleichheit gilt genau bei Erfülltsein von (B_2).* Dieser Satz zeigt, daß man unter geeigneten Zusatzvoraussetzungen das Ergebnis von Satz 3 auch ohne das Axiom (B_1) bzw. (P_1) gewinnen kann. Daß die Voraussetzung $b_2=1$ in dem De Bruijn-Erdösschen Resultat wesentlich ist, zeigt das folgende triviale Beispiel: $|\mathscr{P}|=n>2$, $|\mathscr{B}|=2$, $|I|=|\mathscr{P} \times \mathscr{B}|-1$. Hier gilt weder (P_1) noch (B_1), aber (B_2) und (P_n) sind erfüllt, ohne daß $b_0=p_0$ ist.

5. Taktische Konfigurationen. Die bisher bewiesenen Sätze haben gezeigt, daß die Axiome (B_1) und (P_1) unter den übrigen (B_i) und (P_j) eine besonders wichtige Stellung einnehmen. Eine endliche Inzidenzstruktur, die (B_1) und (P_1) genügt, heißt *taktische Konfiguration* (MOORE [7], CARMICHAEL [4, chapter 14]); auf diese beschränken wir uns in diesem letzten Paragraphen ausschließlich. Dabei soll noch der triviale Fall ausgeschlossen werden, bei dem jedes Element mit jedem andersartigen inzidiert; wir verlangen also

$$I \neq \mathscr{P} \times \mathscr{B}. \tag{5.1}$$

Aus Satz 2 folgt, daß es zu jeder taktischen Konfiguration $\mathscr{K}$ ein wohlbestimmtes Zahlenpaar (m, n) mit $0 < m < b_0$ und $0 < n < p_0$ derart gibt, daß (B_i) und (P_j) für $i = 1, \ldots, m$ und $j = 1, \ldots, n$ erfüllt sind, aber nicht für $i = m+1, \ldots, b_0$ und $j = n+1, \ldots, p_0$. Dieses Zahlenpaar heißt der *Typ* von $\mathscr{K}$ *Ist* (m, n) *der Typ einer taktischen Konfiguration, so gilt für deren (sodann ganzzahlige) Invarianten:*

$$b_0 > b_1 > \cdots > b_n \quad \textit{und} \quad p_0 > p_1 > \cdots > p_m. \tag{5.2}$$

Beweis. Wegen (3.5) braucht nur die Unmöglichkeit von $b_i = b_{i+1}$ für $0 \leqq i < n$ gezeigt zu werden. Nach dem bei (3.5) bewiesenen folgt aus $b_i = b_{i+1}$, daß für jeden Block X entweder $[X] = p_0$ oder $[X] < i$ gilt. Da taktische Konfigurationen (B_1) genügen, heißt das also $p_1 = p_0$ oder $p_1 < i$. Aber $p_1 = p_0$ bedeutet, daß jeder Punkt mit jedem Block inzidiert, was (5.1) widerspricht, und $p_1 < i$ hat wegen der [nach Satz 1 hier mit dem Gleichheitszeichen anwendbaren] Beziehung (3.6), sowie wegen $b_i > 0$ und $b_{i+1} > 0$ [das folgt aus dem Erfülltsein von (P_i) und (P_{i+1})]

$$b_{i+1}(p_0 - i) = b_i(p_1 - i) < 0,$$

also $p_0 < i$ zur Folge, im Widerspruch zu $i < n < p_0$. Damit ist (5.2) bewiesen.

Nach Satz 1 genügen die b_i und p_j den Beziehungen (3.6) mit dem Gleichheitszeichen; im folgenden werden hauptsächlich die Spezialfälle (3.8) gebraucht. Nach Satz 3 ist $b_0 = p_0$ bei $m > 1$ und $n > 1$, und daraus beweist man mit Satz 1 sofort: *Für taktische Konfigurationen des Typs* (m, n) *mit* $m > 1$ *und* $n > 1$ *gilt* $p_i = b_i$, $i = 0, 1, \ldots, b_0$. Im Laufe der folgenden Untersuchung werden wir hierfür einen kurzen neuen Beweis erhalten.

Die Frage nach den möglichen Typen taktischer Konfigurationen wird durch den folgenden Satz beantwortet:

Satz 4. *Ein Paar* (m, n) *von natürlichen Zahlen ist genau dann der Typ einer taktischen Konfiguration, wenn* $m = 1$ *oder* $n = 1$ *oder* $m = n$ *ist.*

Wir zeigen zuerst, daß es zu allen $(1, n)$, $(n, 1)$, (n, n) mit $n = 1, 2, \ldots$ taktische Konfigurationen dieser Typen gibt. Es sei $\mathscr{P}$ eine Menge von N Elementen, $N > n$, und $\mathscr{B}$ die Menge aller n-elementigen Untermengen von $\mathscr{P}$. Weiter sei $p\,I\,B$ für $p \in \mathscr{P}$ und $B \in \mathscr{B}$ gleichbedeutend mit $p \in B$. Man überzeugt sich leicht, daß die so erklärte Inzidenzstruktur eine taktische Konfiguration vom Typ $(1, n)$ oder (n, n) ist, je nachdem, ob $n < N - 1$ oder $n = N - 1$ ist.

Für $n < N-1$ erhält man durch Dualisierung auch eine taktische Konfiguration vom Typ $(n, 1)$. Also sind die behaupteten Bedingungen für m, n hinreichend für die Existenz taktischer Konfigurationen vom Typ (m, n).

Die soeben konstruierten taktischen Konfigurationen sind natürlich nicht sehr interessant: jede von ihnen hat die volle symmetrische Gruppe von $\mathscr{P}$ zur Automorphismengruppe (Automorphismen sind Blöcke auf Blöcke abbildende Punktpermutationen). Der folgende Satz zeigt, daß es im Falle des symmetrischen Typs (n, n) mit $n > 2$ keine anderen taktischen Konfigurationen als diese geben kann:

Satz 5. *Für $n > 2$ sind alle taktischen Konfigurationen des Typs (n, n) isomorph; es gilt $b_i = p_i = n - i + 1$ $(i = 0, \ldots, n)$.*

Der Beweis dieses Satzes ergibt sich, ebenso wie der für die Notwendigkeit der Bedingungen von Satz 4 für m und n aus dem folgenden Hilfssatz:

Lemma 2. *Die folgenden Eigenschaften einer taktischen Konfiguration $(\mathscr{P}, \mathscr{B}, I)$ vom Typ (m, n) sind gleichbedeutend:*

(a) $m > 1$ *und* $n > 2$,

(b) $m > 2$ *und* $n > 1$,

(c) $m, n > 1$, $|\mathscr{B}| > 3$, *und jeder Block inzidiert mit wenigstens einem von zwei beliebig vorgebbaren Punkten,*

(d) $m, n > 1$, $|\mathscr{P}| > 3$, *und jeder Punkt inzidiert mit wenigstens einem von zwei beliebig vorgegbaren Blöcken,*

(e) $|\mathscr{B}| > 3$, $|\mathscr{P}| > 3$, *und zu jedem Element existiert genau ein nichtinzidentes.*

(f) $|\mathscr{B}| = |\mathscr{P}| > 3$ *und* $m = n = |\mathscr{P}| - 1$.

Vor dem Beweis leiten wir aus Lemma 2 die Sätze 4 und 5 her. Eine taktische Konfiguration vom Typ (m, n) mit $m, n > 1$ ist entweder vom Typ $(2, 2)$, oder eine der Bedingungen (a), (b) ist erfüllt. Aus der Äquivalenz von (a) und (b) mit (f) folgt dann $m = n = p_0 - 1 = b_0 - 1$, und das beweist einerseits Satz 4, andererseits ergibt sich $b_i = p_i = n - i + 1$ aus (5.2) und der Ganzzahligkeit der p_i, b_i $(i = 1, \ldots, n)$, woraus Satz 5 folgt. Damit ist alles auf den Beweis von Lemma 2 zurückgeführt. Diesen Beweis führen wir nach dem Schema

$$\text{(f)} \begin{cases} \nearrow \text{(a)} \longrightarrow \text{(c)} \\ \searrow \text{(b)} \longrightarrow \text{(d)} \end{cases} \begin{cases} \searrow \\ \nearrow \end{cases} \text{(e)} \longrightarrow \text{(f)}.$$

Daß (a) und (b) aus (f) folgen, ist trivial, und daß (f) eine Folge von (e) ist, ergibt sich daraus, daß die Blöcke wegen (e) umkehrbar eindeutig den $(p_0 - 1)$-elementigen Punktmengen zugeordnet sind und die Konfiguration dadurch isomorph auf die beim Beweis des Hinreichens der Bedingungen von Satz 4 konstruierte Konfiguration vom Typ (n, n) mit $n = p_0 - 1$ und $N = p_0$ abgebildet wird, für die (f) natürlich gilt. Also bleibt (aus Dualitätsgründen) nur die Richtigkeit der Implikationen (a) $\rightarrow$ (c) und (c) $\rightarrow$ (e) zu zeigen. Diesen Beweisen schicken wir noch einen Hilfssatz voraus:

In einer taktischen Konfiguration vom Typ (m, n) *seien* $\mathfrak{B}$ *und* $\mathfrak{p}$ *eine Block- und eine Punktmenge mit* $0<|\mathfrak{B}|<m$, $0<|\mathfrak{p}|<n$ *und* $(\mathfrak{p}\times\mathfrak{B})\cap I=\emptyset$. *Dann gilt*

(5.3) $\qquad b_i p_{j+1}=p_j b_{i+1}$ *für* $1\leqq i\leqq|\mathfrak{p}|$ *und* $1\leqq j\leqq|\mathfrak{B}|$.

Beweis. $\mathfrak{B}$ und $\mathfrak{p}$ mit den verlangten Eigenschaften seien vorgegeben. Ist dann $1\leqq i\leqq|\mathfrak{p}|$ und $1\leqq j\leqq|\mathfrak{B}|$, so enthält $\mathfrak{p}$ eine Untermenge $\mathfrak{x}$ mit $|\mathfrak{x}|=i$ und $\mathfrak{B}$ eine Untermenge $\mathfrak{X}$ mit $|\mathfrak{X}|=j$, und aus $(\mathfrak{p}\times\mathfrak{B})\cap I=\emptyset$ folgt auch $(\mathfrak{x}\times\mathfrak{X})\cap I=\emptyset$, d.h. kein Punkt von $\mathfrak{x}$ ist mit einem Block von $\mathfrak{X}$ inzident. Folglich ist $[\mathfrak{X}, \mathfrak{x}]=[\mathfrak{x}, \mathfrak{X}]=0$ (vgl. 2), und Lemma 1, angewandt auf $\mathfrak{x}$ und $\mathfrak{X}$, ergibt die Beziehung

$$\sum_{X\,I\,\mathfrak{x}}[\mathfrak{X}\cup\{X\}]=\sum_{x\,I\,\mathfrak{X}}[\mathfrak{x}\cup\{x\}].$$

Aber aus $|\mathfrak{X}|<m$ und $|\mathfrak{x}|<n$ folgt die Gültigkeit von (P_i) und (P_{i+1}) sowie von (B_j) und (B_{j-1}). Also ist jeder Summand links gleich p_{j+1} und rechts b_{i+1}, und die Behauptung (5.3) folgt.

(Zwischenbemerkung. Wegen (5.1) gibt es sicher ein nichtinzidentes Paar (p, B). Die Mengen $\mathfrak{p}=\{p\}$ und $\mathfrak{B}=\{B\}$ erfüllen sodann bei $m, n>1$ die Voraussetzungen des soeben bewiesenen Hilfssatzes mit $i=j=1$. Also liefert (5.3) für diesen Fall die Gleichung

$$b_1 p_2=p_1 b_2.$$

Substituiert man nun p_2 und b_2 gemäß (3.8), so ergibt sich nach trivialen Umformungen

$$b_0+p_1=b_1+p_0.$$

Das ist mit der ersten Formel (3.8) nur dann verträglich, wenn die Zahlenpaare (p_0, b_1) und (b_0, p_1) sich höchstens in der Reihenfolge unterscheiden. Aber $p_0=p_1$ widerspricht (5.2), also folgt $p_0=b_0$, daraus mit (3.8) weiter $p_1=b_1$, und nun ergibt sich aus den Formeln (3.6) [mit dem Gleichheitszeichen], daß $p_i=b_i$ für alle i ist. Dies ist der angekündigte zweite Beweis für die Tatsache, daß aus $m, n>1$ immer $p_i=b_i$ folgt. Man vergleiche hierzu MAJUMDAR [6, theorem 1].)

Wir führen jetzt den Beweis von Lemma 2 zu Ende.

Aus (a) *folgt* (c): Aus $n>2$ ergibt sich sofort $|\mathfrak{B}|>3$ wegen (5.1). Sei $\mathfrak{p}$ eine aus zwei Punkten bestehende Punktmenge. Gäbe es einen Block B derart, daß kein Punkt von $\mathfrak{p}$ auf B liegt, so würden $\mathfrak{p}$ und $\mathfrak{B}=\{B\}$ die Voraussetzungen des Hilfssatzes erfüllen. Also müßte (5.3) für die Indexpaare $(i, j)=(1, 1)$ und $(i, j)=(2, 1)$ gelten:

$$b_1 p_2=p_1 b_2 \quad\text{und}\quad b_2 p_2=p_1 b_3.$$

Hieraus folgt aber bei Anwendung von (3.6) [mit dem Gleichheitszeichen]

$$\frac{p_1-1}{p_0-1}=\frac{b_2}{b_1}=\frac{p_2}{p_1}=\frac{b_3}{b_2}=\frac{p_1-2}{p_0-2},$$

also würde sich $p_0 = p_1$ ergeben, im Widerspruch zu (5.2). Also gibt es keinen solchen Block B, und (c) ist bewiesen.

Aus (c) *folgt* (e): Durch zwei verschiedene Punkte gehen je b_1 Blöcke, und b_2 gehen durch beide Punkte. Da nach (c) jeder Block durch wenigstens einen der beiden Punkte geht, folgt also

$$b_0 = 2b_1 - b_2.$$

Substituiert man nun wieder b_2 gemäß (3.8) und ersetzt p_0 durch b_0 und p_1 durch b_1, was wegen der Voraussetzung $m, n > 1$ in (c) nach der obigen Zwischenbemerkung erlaubt ist, so ergibt sich nach leichten Umformungen

$$(b_0 - b_1)(b_0 - b_1 - 1) = 0,$$

und daraus folgt wegen (5.2), daß $b_0 = b_1 + 1$ sein muß. Also ist auch $p_0 = p_1 + 1$, und damit ist (e) bewiesen.

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Ein Kriterium für die Distributivität des Untergruppenverbandes einer abelschen Operatorgruppe

Von

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Eine Gruppe heißt *D-Gruppe*, wenn der Verband ihrer Untergruppen distributiv ist. Suzuki [4, Theorem 2] hat bewiesen:

A group G is a D-group, if and only if every finite set of elements in G generates a cyclic subgroup.

Andererseits habe ich gezeigt [3, §3, Satz 2]:

Sei g eine abelsche Gruppe mit einem Ringe Ω von Operatoren; Ω enthalte ein Einselement und dieses operiere identisch in g. Dann sind die folgenden Eigenschaften von (g, Ω) äquivalent:

(1) *Der Verband der Ω-Untergruppen von g ist distributiv.*

(2) *Für je zwei Ω-Untergruppen a, b von g und jedes Element $\gamma \in a \cup b$ gilt $(a:\gamma) \cup (b:\gamma) = \Omega$ (im Verband der Rechtsideale von Ω)*[1].

(3) *Zu beliebigem Ω-Untergruppenpaare a, b von g gibt es keinen nicht trivialen Homomorphismus zwischen $a-(a \cap b)$ und $b-(a \cap b)$.*

(4) *Für je zwei Ω-Untergruppen a, b von g gilt: Aus $a-(a \cap b) \cong b-(a \cap b)$ folgt $a=b$.*

In dieser Arbeit nun soll eine Verallgemeinerung des Distributivitätskriteriums von Suzuki auf abelsche Operatorgruppen bewiesen werden (Satz 1); sie stellt sich den genannten vier äquivalenten Eigenschaften der Operatorgruppe (g, Ω) als fünfte an die Seite. Die Spezialisierung auf den Fall, daß g die additive Gruppe eines Ringes ist, ergibt in §2 ein Kriterium dafür, daß jeder zyklische Modul eines Ringes distributiven Untermodulverband besitzt.

§ 1. Gegeben sei eine additiv geschriebene abelsche Gruppe g mit einer Menge Ω von Rechtsoperatoren. Für die Faktorgruppe zweier Untergruppen a, b von g mit $a > b$ schreiben wir $a-b$. Ferner werde der Verband aller Ω-Untergruppen von g mit $V(g, \Omega)$ bezeichnet. Eine Ω-Gruppe z heiße *zyklisch*, wenn $z = \zeta\Omega$ für ein Element $\zeta \in z$ gilt.

[1] $a:\gamma$ bezeichnet den „Quotienten" $\{\omega; \omega \in \Omega, \gamma\omega \in a\}$.

Ziel dieses Paragraphen ist der folgende

1.1. SATZ 1. *Der Verband $V(g, \Omega)$ ist genau dann distributiv, wenn gilt: Zu jedem Elementepaar $\alpha, \beta \in g$ ist die Faktorgruppe $(\alpha\Omega \cup \beta\Omega) - (\alpha\Omega \cap \beta\Omega)$ zyklisch mit der erzeugenden Restklasse $(\alpha+\beta)+(\alpha\Omega \cap \beta\Omega)$.*

Dem Beweise dienen die beiden Lemmata 1.3 und 1.5.

Wie SUZUKI in [4] erklären wir zunächst:

1.2. DEFINITION. Sei A ein Verband. Ein Elementepaar $a, b \in A$ heißt *distributives Paar in A*, wenn für alle $c \in A$ gilt $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$.

1.3. LEMMA. *Ein Untergruppenpaar $a, b \in V(g, \Omega)$ bildet genau dann ein distributives Paar in diesem Verbande, wenn es zu jedem Element $\gamma \in a \cup b$ ein Element $\mu \in \Omega$ gibt mit $\gamma\mu \in a$, $\gamma - \gamma\mu \in b$.*

BEWEIS. 1. a, b sei distributives Paar und $\gamma \in a \cup b$. Speziell für $c = \gamma\Omega$ gilt dann $\gamma\Omega = \gamma\Omega \cap (a \cup b) = (\gamma\Omega \cap a) \cup (\gamma\Omega \cap b)$. Also hat γ die Darstellung $\gamma = \gamma\mu + \gamma\mu^*$ mit $\gamma\mu \in \gamma\Omega \cap a$, $\gamma\mu^* \in \gamma\Omega \cap b$. Es folgt $\gamma\mu \in a$, $\gamma - \gamma\mu = \gamma\mu^* \in b$.

2. Die angegebene Bedingung sei für a, b aus A erfüllt. Für jedes $x \in A$, $x < a \cup b$, gilt dann $x = (x \cap a) \cup (x \cap b)$. Denn jedes $\xi \in x$ hat die Form $\xi = \xi\mu + \xi - \xi\mu$, mit $\xi\mu \in x \cap a$, $\xi - \xi\mu \in x \cap b$.

Ist $c \in A$ beliebig, so hat man also für $x = c \cap (a \cup b)$ die Beziehung

$$c \cap (a \cup b) = x = (x \cap a) \cup (x \cap b) = \big(c \cap (a \cup b) \cap a\big) \cup \big(c \cap (a \cup b) \cap b\big) = (c \cap a) \cup (c \cap b).$$

1.4. ZUSÄTZE. Offensichtlich lassen sich die Rollen von a und b in 1.3 vertauschen. Es ergibt sich so die Form des Distributivitätskriteriums „Zu jedem Element $\gamma \in a \cup b$ gibt es ein Element $\nu \in \Omega$ mit $\gamma\nu \in b$, $\gamma - \gamma\nu \in a$“; oder insgesamt die Form: *Zu jedem Element $\gamma \in a \cup b$ gibt es Elemente $\mu, \nu \in \Omega$ mit $\gamma\mu \in a$, $\gamma\nu \in b$; $\gamma - \gamma\mu \in b$, $\gamma - \gamma\nu \in a$.*

Ist z.B. Ω ein Ring mit identisch operierendem Einselement, so kann man für ν das Element $1 - \mu$ nehmen.

Aus der Tatsache, daß die Elemente aus $a \cup b$ genau die Elemente der Gestalt $\alpha + \beta$ mit $\alpha \in a$, $\beta \in b$ sind, folgert man sofort, daß sich das Kriterium in 1.3 auch in die beiden Formen kleiden läßt:

Zu jedem Paar von Elementen $\alpha \in a$, $\beta \in b$ gibt es ein Element $\nu \in \Omega$ mit $\alpha\nu \in \mathfrak{a} \cap \mathfrak{b}$, $\beta - \beta\nu \in \mathfrak{a} \cap \mathfrak{b}$.

Zu jedem Paar von Elementen $\alpha \in a$, $\beta \in b$ gibt es Elemente $\mu, \nu \in \Omega$ mit $\alpha\nu \in b$, $\beta\mu \in a$, $\alpha - \alpha\mu \in b$, $\beta - \beta\nu \in a$.

Wird im Folgenden angenommen, daß Ω ein Ring mit Einselement ist, so sei stets vorausgesetzt, daß dieses in g identisch operiere.

1.5. LEMMA. *Sei $a = \alpha\Omega$, $b = \beta\Omega$ ein Paar zyklischer Ω-Untergruppen von g; sei $\bar{g} = g - (a \cap b)$, entsprechend $\bar{a}$, $\bar{b}$.*

(a) *Ist $\bar{a}$, $\bar{b}$ distributives Paar in $\bar{g}$, so ist $\bar{a} \cup \bar{b} = \overline{a \cup b}$ zyklisch mit der erzeugenden Restklasse $\overline{\alpha + \beta}$.*

(b) *Ω sei ein Ring mit Einselement und es sei in Ω jedes Rechtsideal zweiseitig. Ist dann $\bar{a}$, $\bar{b}$ zyklische Ω-Gruppe, so wird $\bar{a}\cup\bar{b}$ von der Restklasse $\overline{\alpha+\beta}$ erzeugt und die Gruppen a, b bilden ein distributives Paar in $V(g,\Omega)$.*

Beweis. Zu (a): Aus den Voraussetzungen folgt die Existenz von Elementen $\mu,\nu\in\Omega$ mit $\bar{\alpha}\nu\in\bar{a}$, $\bar{b}=\bar{0}$, $\bar{\beta}-\bar{\beta}\nu=\bar{0}$, $\bar{\alpha}-\bar{\alpha}\mu=\bar{0}$, $\bar{\beta}\mu=\bar{0}$. Also gilt $\overline{(\alpha+\beta)}\mu=\bar{\alpha}\mu+\bar{\beta}\mu=\bar{\alpha}$, $\overline{(\alpha+\beta)}\nu=\bar{\beta}$; $\overline{\alpha+\beta}$ erzeugt ganz $\bar{a}\cup\bar{b}$.

Zu (b): $\bar{a}\cup\bar{b}$ sei zyklisch.

1. Wir zeigen zunächst, daß man als erzeugende Restklasse $\overline{\alpha+\beta}$ nehmen kann.

Sei $\bar{a}\cap\bar{b}=\bar{\gamma}\Omega$ mit $\bar{\gamma}=\bar{\alpha}'-\bar{\beta}'$ ($\bar{\alpha}'\in\bar{\alpha}\Omega$, $\bar{\beta}'\in\bar{\beta}\Omega$). Es ist also $\bar{\alpha}'=\bar{\alpha}\omega$ für ein $\omega\in\Omega$, andererseits $\bar{\alpha}=\bar{\alpha}'\iota+\bar{\beta}'\iota$ für ein $\iota\in\Omega$, woraus — da die Summe direkt ist — folgt $\bar{\beta}'\iota=\bar{0}$, also $\bar{\alpha}=\bar{\alpha}'\iota$. Analog erschließt man $\bar{\beta}=\bar{\beta}'\varkappa$ für ein $\varkappa\in\Omega$. Es folgt

$$\bar{\alpha}=\bar{\alpha}\omega\iota. \tag{1}$$

Weiter ist $\bar{\beta}\omega\iota=\bar{\beta}'\varkappa\omega\iota$. Das Rechtsideal $\iota\Omega$ ist zweiseitig; es gibt also ein $\xi\in\Omega$ mit $(\varkappa\omega)\iota=\iota\xi$. Hiermit hat man $\bar{\beta}\omega\iota=\bar{\beta}'\iota\xi=\bar{0}$ (da $\bar{\beta}'\iota=\bar{0}$), zusammen mit (1) also $\overline{(\alpha+\beta)}\Omega\ni\overline{(\alpha+\beta)}\omega\iota=\bar{\alpha}\omega\iota+\bar{\beta}\omega\iota=\bar{\alpha}$. Analog erschließt man $\bar{\beta}\in\overline{(\alpha+\beta)}\Omega$.

2. Mit $\mu=\omega\iota$ ist also $\overline{(\alpha+\beta)}\mu=\bar{\alpha}$, $\overline{(\alpha+\beta)}(1-\mu)=\bar{\alpha}+\bar{\beta}-\bar{\alpha}\mu-\bar{\beta}\mu=\bar{\beta}$, und so (wegen der Voraussetzung über Rechtsideale)

$$\overline{(a\cup b)}\,\mu=\overline{(\alpha+\beta)}\,\Omega\,\mu<\overline{(\alpha+\beta)}\,\mu\,\Omega=\bar{a}$$

$$\overline{(a\cup b)}\,(1-\mu)=\overline{(\alpha+\beta)}\,\Omega\,(1-\mu)<\overline{(\alpha+\beta)}\,(1-\mu)\,\Omega=\bar{b},$$

Man hat also

$$(a\cup b)\,\mu<a\cup(a\cap b)=a$$

$$(a\cup b)\,(1-\mu)<b\cup(a\cap b)=b,$$

woraus sich nach 1.4 die Behauptung ergibt.

Beweis von Satz 1.

1. Mit einem Verbande A ist auch jeder Teilverband von A distributiv, mit $V(g,\Omega)$ also auch $V\big(g-(\alpha\Omega\cap\beta\Omega),\Omega\big)$. Nach 1.5 (a) folgt, daß die in Satz 1 genannte Bedingung für die Distributivität von $V(g,\Omega)$ notwendig ist.

2. Sei umgekehrt $V(g,\Omega)$ nicht distributiv. Nach [3, Satz 2, (3.5)] enthält $V(g,\Omega)$ zwei verschiedene Untergruppen a, b von g mit $a-(a\cap b)\underset{\Omega}{\cong}b-(a\cap b)$[2]; sei σ ein Ω-Isomorphismus von $\bar{a}=a-(a\cap b)$ auf $\bar{b}=b-(a\cap b)$. Sei endlich α ein nicht in b gelegenes Element von a; es ist $\bar{\alpha}^{\sigma}\in\bar{b}$.

Angenommen, für ein $\omega\in\Omega$ gälte $\bar{\alpha}=(\bar{\alpha}+\bar{\alpha}^{\sigma})\omega$. Dann hätte man also $\bar{\alpha}=\bar{\alpha}\omega+(\bar{\alpha}\omega)^{\sigma}$ (da σ ein Ω-Isomorphismus ist), woraus wegen der Direktheit

[2]) Satz 2 in Menzel [3] ist noch unter der Voraussetzung formuliert, daß Ω ein Ring mit Einselement sei. Jedoch ist — wie sich aus dem Beweise sofort ergibt — dies für die hier verwendete Teilaussage nicht von Belang.

der Summe $\bar{\alpha}\Omega \cup \bar{\alpha}^{\sigma}\Omega$ folgt $\bar{\alpha} = \bar{\alpha}\omega$, $(\bar{\alpha}\omega)^{\sigma} = \bar{0}$. Aber σ ist Isomorphismus, und so folgt aus der letzten Beziehung $\bar{\alpha}\omega = \bar{0}$, $\bar{\alpha} = \bar{0} + \bar{0} = \bar{0}$ entgegen $\alpha \notin b$. Es ist $\bar{\alpha} \neq (\bar{\alpha} + \bar{\alpha}^{\sigma})\omega$ für alle $\omega \in \Omega$.

Ist β ein Vertreter der Restklasse $\bar{\alpha}^{\sigma}$, so folgt für alle $\omega \in \Omega$, daß $(\alpha+\beta)\omega \not\equiv \alpha$ modulo $a \cap b$, also erst recht $(\alpha+\beta)\omega \not\equiv \alpha$ modulo $\alpha\Omega \cap \beta\Omega$ ist. Also ist es unmöglich, daß die Restklasse $(\alpha+\beta) + (\alpha\Omega \cap \beta\Omega)$ die ganze Gruppe $(\alpha\Omega \cup \beta\Omega) - (\alpha\Omega \cap \beta\Omega)$ erzeugt.

1.6. FOLGERUNG 1. *Ω sei ein Ring mit Einselement und in Ω sei jedes Rechtsideal zweiseitig. Dann ist die Distributivität des Verbandes $V(g, \Omega)$ aller Ω-Untergruppen der Gruppe g gleichbedeutend damit, daß zu je zwei Elementen $\alpha, \beta \in g$ die Faktorgruppe $(\alpha\Omega \cup \beta\Omega) - (\alpha\Omega \cap \beta\Omega)$ zyklisch ist.*

BEWEIS. Daß die Bedingung notwendig ist, ist klar. Umgekehrt sieht man leicht, daß der Untergruppenverband $V(g, \Omega)$ bereits dann distributiv ist, wenn je zwei *zyklische* Untergruppen a, b in $V(g, \Omega)$ ein distributives Paar bilden. Aus 1.5 folgt nun sofort, daß die Bedingung auch hinreicht.

Wir haben weiter (für abelsche Gruppen) die folgende leichte Verallgemeinerung des in der Einleitung genannten Distributivitätskriteriums von SUZUKI [4]:

1.7. FOLGERUNG 2. *Der Operatorbereich Ω der Gruppe g sei ein Hauptidealring mit Einselement, und in Ω sei jedes Rechtsideal zweiseitig. Dann ist die Distributivität von $V(g, \Omega)$ äquivalent damit, daß je endlich viele Elemente von g eine zyklische Untergruppe erzeugen.*

BEWEIS. 1. Aus 1.6 ergibt sich, daß die Bedingung hinreicht.

2. Sei $V(g, \Omega)$ distributiv. Da Ω Hauptidealring ist, ist jede endlich erzeugte Ω-Untergruppe von g direkte Summe endlich vieler zyklischer Ω-Untergruppen von g.

Angenommen, es gäbe eine nicht zyklische direkte Summe endlich vieler zyklischer Ω-Untergruppen. Dann ergibt sich sofort auch die Existenz einer nicht zyklischen direkten Summe zweier Summanden, etwa $\alpha\Omega \oplus \beta\Omega$. Nach 1.4 existiert ein Element $\nu \in \Omega$ mit $\alpha\nu = 0$, $\beta\nu = \beta$. Es folgt $(\alpha+\beta)\nu = \beta$, $(\alpha+\beta)(1-\nu) = \alpha$; α und β liegen in $(\alpha+\beta)\Omega$. Also ist $\alpha\Omega \oplus \beta\Omega$ zyklisch entgegen Annahme.

§ 2. Wir spezialisieren die Ergebnisse des § 1 auf den Fall, daß Ω ein Ring mit Einselement, g die additive Gruppe Ω^{+} des Ringes ist. $V(g, \Omega) = V(\Omega^{+}, \Omega)$ ist dann der Verband der Rechtsideale von Ω. Unter einem *Modul* des Ringes Ω verstehen wir stets einen *Rechtsmodul*.

Sei A ein Verband. Wie in [2] werde ein Element $i \in A$ *voll $\cup$-irreduzibel* genannt, wenn es nicht als Verbindung echt früherer Verbandselemente dargestellt werden kann. A heiße *glatt*, wenn jedes Element von A Verbindung voll $\cup$-irreduzibler Elemente ist. Weiter bezeichnen wir den Verband A als *$\uparrow$-atomar* (vgl. [1], S. 38), wenn es zu je zwei Elementen $a, b \in A$ mit $b < a$, $b \neq a$, obere Nachbarn von b unter a gibt.

In [3, Satz 4] wurde bewiesen: *Der Verband $V(g, \Omega)$ der Ω-Untergruppen von g sei glatt, distributiv und $\uparrow$-atomar; ferner sei Ω ein Ring mit Einselement. Ist dann $a \in V(g, \Omega)$ und η ein Ω-Homomorphismus von a in g, so gilt $a^{\eta} \subset a$.*

Durch Spezialisierung der Gruppe g zur additiven Gruppe eines Ringes Ω mit Einselement folgt hieraus: *Ω sei ein Ring mit Einselement. Der Verband der Rechtsideale von Ω sei glatt, distributiv und $\uparrow$-atomar. Dann ist jedes Rechtsideal in Ω zweiseitig.*

Hiermit nun läßt sich beweisen

2.1. Satz 2. *Ω sei ein Ring mit Einselement, der Verband der Rechtsideale von Ω sei glatt und $\uparrow$-atomar. Dann sind die folgenden Eigenschaften von Ω äquivalent:*

(1) *Jeder zyklische Ω-Modul hat distributiven Untermodulverband.*

(2) *Der Verband der Rechtsideale von Ω ist distributiv.*

(3) *Es gilt*

(a) *Alle Rechtsideale in Ω sind zweiseitig.*

(b) *Zu jedem Paar P, Σ von (Rechts-)Hauptidealen in Ω ist $(\mathsf{P} \cup \Sigma)/(\mathsf{P} \cap \Sigma)$ im Restklassenring $\Omega/(\mathsf{P} \cap \Sigma)$ Hauptideal.*

Beweis. 1. Da Ω ein Einselement besitzt, stimmt die Gesamtheit der zyklischen Ω-Moduln überein mit der Gesamtheit der Ω-homomorphen Bilder von Ω^+. Die Distributivität des Verbandes $V(\Omega^+, \Omega)$ der Rechtsideale von Ω vererbt sich auf jeden Verband $V(\Omega^{+\eta}, \Omega)$ (wo η ein Ω-Homomorphismus ist); genau dann hat jeder zyklische Ω-Modul distributiven Untermodulverband, wenn $V(\Omega^+, \Omega)$ distributiv ist.

2. Seien P, Σ Rechtsideale in Ω. Wir bezeichnen durch Überstreichung Restklassen- bzw. Faktorgruppenbildung nach $\mathsf{P} \cap \Sigma$ in der Ω-Gruppe Ω^+.

Jedes Rechtsideal in Ω sei zweiseitig. Dann ist $\Omega/\mathsf{P} \cap \Sigma$ ein Ring, den wir mit $\overline{\Omega}$ bezeichnen wollen, und die Faktorgruppe $\overline{\Omega}^+$ ist $\overline{\Omega}$-Modul vermöge der Definition $\bar{\xi}\,\bar{\omega} = \bar{\xi}\omega$ $(\bar{\xi} \in \overline{\Omega}^+, \omega \in \Omega, \bar{\omega} \in \overline{\Omega})$; es ist $\overline{\Omega^+} = \overline{\Omega}^+$. Jedes Rechtsideal in $\overline{\Omega}$ ist zweiseitig. Die $\overline{\Omega}$-Gruppen in $\overline{\Omega}^+$ sind die Rechtsideale, also die Ideale, in $\overline{\Omega}$. Für ein Rechtsideal T in Ω folgt die Äquivalenz der Eigenschaften „$\overline{\mathsf{T}}$ ist zyklisch als Ω-Modul", „$\overline{\mathsf{T}}$ ist Rechtshauptideal in $\overline{\Omega}$" und „$\overline{\mathsf{T}}$ ist Hauptideal in $\overline{\Omega}$".

3. Aus 2. ergibt sich, daß die Äquivalenz der Eigenschaften (2) und (3) in Satz 2 der folgenden Aussage gleichwertig ist:

Der Verband $V(\Omega^+, \Omega)$ ist genau dann distributiv, wenn gilt

(a′) Jedes Rechtsideal in Ω ist zweiseitig.

(b′) Zu je zwei zyklischen Ω-Untermoduln P, Σ des Ω-Moduls $\overline{\Omega}^+$ ist $(\mathsf{P} \cup \Sigma) - (\mathsf{P} \cap \Sigma)$ zyklisch.

Die Notwendigkeit dieser Bedingungen folgt sofort aus den zu Beginn dieses Paragraphen zitierten Ergebnissen von [3] und aus Satz 1. Aus 1.6 ergibt sich, daß sie auch hinreichend sind. Hiermit ist Satz 2 bewiesen.

2.2. ZUSATZ. Aus dem Beweis ergibt sich, daß Satz 2 dadurch ein klein wenig verschärft werden kann, daß man Eigenschaft (3) (b) ersetzt durch die genauere Aussage

Für je zwei Hauptideale P, Σ *in* Ω *ist im Restklassenring* $\Omega/(\mathsf{P}\cap\Sigma)$ *das Ideal* $(\mathsf{P}\cup\Sigma)/(\mathsf{P}\cap\Sigma)$ *Hauptideal und wird von einer Restklasse* $\bar{\tau}\in\Omega/(\mathsf{P}\cap\Sigma)$ *in der Form* $\bar{\tau}(\Omega/\mathsf{P}\cap\Sigma)$ *erzeugt.*

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Groups with regular automorphisms of order four

By
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1. Introduction

An automorphism of a group G is called regular if it moves every element of G except the identity. Burnside proved that a finite group G has a regular automorphism of order two if and only if G is an abelian group of odd order, and then the only such automorphism maps every element onto its inverse ([2], p. 230). More recently several authors considered the question: what groups can admit regular automorphisms of order a prime p? B. H. Neumann [8] and M. Nagata [7] extended the original theorem about the case $p=2$ to various classes of infinite groups. The case $p=3$ appeared in the second edition of Burnside's book ([3], pp. 90–91), and this theorem was also generalized by Neumann [9]. For arbitrary p, the results so far can be summarized as follows. If a locally finite or locally nilpotent group G has a regular automorphism of prime order p, then G is nilpotent and its class is bounded in terms of p (G. Higman [4] and J. Thompson [10]).

In this paper we consider groups which admit regular automorphisms of order four. For this case one cannot obtain results like those quoted above. Finite nilpotent groups of every class have regular automorphisms of order four; hence locally nilpotent but non-nilpotent groups have them too. There are also non-nilpotent finite soluble groups, both metabelian and non-metabelian ones, that admit such automorphisms. (Examples are given in the last section of the paper.) What we can prove is that *if G is either a locally nilpotent or a periodic locally soluble group, and if G has a regular automorphism of order four, then the second derived group of G is contained in the centre of G.**) The examples just mentioned show this conclusion to be in a sense the best possible.

An outline of the proof is the following.

Let G be an arbitrary group with a regular automorphism α of order four, and let us consider the action of α^2. It is clear that α^2 need not be regular: the elements of G fixed by α^2 form a subgroup $T(G)$. The restriction of α to this subgroup is a regular automorphism of order two and so, under any of the conditions that we later adopt, $T(G)$ is abelian and its elements are all inverted by α. It is more difficult to see what happens to the elements

*) [*Added in proof*, 31. 12. 1960.] Dr. B. Huppert has kindly informed me of a new theorem of D. Gorenstein and I. N. Herstein, according to which a finite group with a regular automorphism of order four is always soluble. Hence in our result the condition "periodic locally soluble" could be replaced by "locally finite".

on which α^2 acts as if it were regular, that is, which are inverted by α^2. The set $S(G)$ formed by these elements need not be a subgroup. However, if we assume that every element of G has a unique square root in G, then G is the product of $T(G)$ and $S(G)$. This then implies that the subgroup $H(G)$ generated by $S(G)$ contains the derived group of G.

Here we turn to the investigation of $H(G)$. Since $H(G)$ may coincide with G, and in any case α can be restricted to $H(G)$, we may consider $H(G)$ without further reference to G. Thus let H be an arbitrary group which has a regular automorphism α of order four, such that the elements inverted by α^2 generate H. Assuming that H is locally nilpotent, commutator computations give that H is nilpotent of class at most three. Then we prove (by induction on the order of H) that the same conclusion holds if instead of local nilpotency we assume finiteness and solubility, and hence also if H is taken to be only periodic and locally soluble.

Returning to G with the condition on roots, we show that if $H(G)$ is nilpotent, then the second derived group of G is contained in the centre of G. Here again some computational work is involved, but after this the main theorems follow immediately. Namely, let G be an arbitrary group with a regular automorphism α of order four. If G is periodic and locally soluble, then the condition on roots is automatically satisfied. If G is locally nilpotent, then it can be embedded in a locally nilpotent group which satisfies the condition on roots and admits an extension of α as a regular automorphism of order four. So the previous theorems can be applied, and in either case we get that the second derived group of G is contained in the centre of G.

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2. Notations and quotations

We follow the terminology and notation of Kurosh [5], apart from the exceptions and with the additions given in the first part of this section.

Let G be any group; then $H \subseteq G$ means that H is a subset, and $H \leqq G$ means that H is a subgroup of G. When H is a normal subgroup of G and a, b are elements of G belonging to the same coset of H, then the congruence $a \equiv b \bmod H$ may be used as a convenient way of recording this situation. We need the usual abbreviations $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$ in the following extended forms. For two subsets A, B of G, the symbol A^B denotes the set of all elements a^b with $a \in A$, $b \in B$. The n-th commutator $[h_1, \ldots, h_n]$ of the elements $h_1, \ldots, h_n$ of G is defined recursively by $[h_1, \ldots, h_n] = [[h_1, \ldots, h_{n-1}], h_n]$. If $H_1, \ldots, H_n$ are subsets of G, then $[H_1, \ldots, H_n]$ stands for the subgroup generated by all the commutators $[h_1, \ldots, h_n]$ with $h_1 \in H_1, \ldots, h_n \in H_n$.

The derived length of a soluble group G is the smallest integer n such that $G^{(n)}$ — the n-th derived group of G — is the unit subgroup E. Metabelian groups are the soluble groups of derived length two. We write the lower central chain of G as $G=\Gamma_1(G)\geqq\Gamma_2(G)\geqq\cdots$, and the upper central chain of G as $E=Z_0(G)\leqq Z_1(G)\leqq\cdots$.

In the course of computations, the following identities are used without further reference:

$$a^b = a\,[a,b],$$

$$[b,a] = [a,b]^{-1},$$

$$[a,b^{-1}] = [b,a]^{b^{-1}},$$

$$[a^{-1},b^{-1}] = [a,b]^{b^{-1}a^{-1}},$$

$$[a\,b,c\,d] = [a,d]^b\,[b,d]\,[a,c]^{b\,d}\,[b,c].$$

Another important identity is

$$[a,b^{-1},c]^b\,[b,c^{-1},a]^c\,[c,a^{-1},b]^a = 1; \tag{2.1}$$

we make a point of quoting it by number whenever it is applied.

Next we collect some definitions and results on groups with unique square roots. We take most of them from BAUMSLAG's thesis [1], but our notation is slightly different and the quotations rarely verbatim: as we use only special cases of his concepts and results, we do not quote them in their full generality. The numbering of the statements follows that in [1]; for instance, (B.5.1) stands for (a possibly simplified form of) (5.1) in [1].

Let G be any group, and $g\in G$. If $x^2=g$ $(x\in G)$, we write $x=g\sigma$ and say that x is a square root of g. If σ is a single-valued function (on the subset of G where it is defined at all), then we call G a Π-group, or write for short that $G\in[\Pi]$. If $G\in[\Pi]$ and if σ is defined on the whole of G, then G is called a Σ-group; for short, $G\in[\Sigma]$.

(B.5.1) *A periodic group G is a Σ-group if and only if G has no element of order two.*

(B.5.3) *If $G\in[\Pi]$, if η is any endomorphism of G, and if g is any element of G such that $g\sigma$ exists, then $(g\eta)\sigma$ also exists and $(g\eta)\sigma=(g\sigma)\eta$.*

(B.5.6) *If $G\in[\Pi]$ and $g,h\in G$, then $[g^{2^n},h^{2^m}]=1$ (n, m non-negative integers) is equivalent to $[g,h]=1$.*

Again, let G be any group. A subset H of G is closed (in G) if $x^2\in H$ $(x\in G)$ implies that $x\in H$. (We usually omit the reference to G as it is always clear from the context which group plays its role.) The intersection of closed subsets is closed. Let $cl(H)$ be the intersection of all closed subgroups of G that contain H; this is the minimal closed subgroup that contains H, and it is called the closure of H in G. A constructive definition of $cl(H)$ is the following. Let H_1 be the subgroup generated by H, ..., let H_{i+1} be the subgroup generated by all x $(\in G)$ such that $x^2\in H_i$, ... Then $cl(H)$ is the union of

$H_1, \ldots, H_i, \ldots$. — If H is normal, characteristic in G, then $cl(H)$ is also normal, characteristic in G, respectively. We shall use $\Delta_i(G)$ to denote the closure of $\Gamma_i(G)$ in G.

Any closed subgroup of a Σ-group is itself a Σ-group.

An ideal is a normal subgroup N of G such that G/N is a Π-group. Every ideal is closed.

(B.9.1) *If H, K are subsets of a Π-group and if $[H, K]=E$, then also $[cl(H), cl(K)]=E$.*

(B.9.2) *All terms of the upper central chain of a Π-group are ideals.*

(B.9.4) *A locally nilpotent group is a Π-group if and only if it has no element of order two.*

(B.9.5) *Any closed normal subgroup of a locally nilpotent group is an ideal.*

(B.10.2) *If G is a Π-group, and if H is an ideal and K a subgroup of G such that $[H, K]=E$ and $cl(K)\geqq H$, then $cl(KH/H)=cl(K)/H$.*

M. Lazard has proved the following statement (in a more general form; [6], pp. 180, 186):

(2.2) *If G is a locally nilpotent group without elements of order two, then there exists a locally nilpotent Σ-group G^* such that* (i) *G^* contains G,* (ii) *to every element g^* of G^* there is a number $n=n(g^*)$ for which $(g^*)^{2^n}$ belongs to G, and* (iii) *every automorphism of G can be extended in one and only one way to an automorphism of G^*.*

Now we turn to automorphisms. Let us repeat that an automorphism α of a group G is called regular if $g\alpha=g$ $(g\in G)$ implies that $g=1$. For brevity we shall say that G is an A_n-group if G has a regular automorphism whose order divides n; sometimes we shall write $(G, \alpha)\in[\mathsf{A}_n]$ to express that α is such an automorphism of G. We shall also use $(G, \alpha)\in[\mathsf{A}_n\Pi]$, $(G, \alpha)\in[\mathsf{A}_n\Sigma]$ if simultaneously $(G, \alpha)\in[\mathsf{A}_n]$ and $G\in[\Pi]$, $(G, \alpha)\in[\mathsf{A}_n]$ and $G\in[\Sigma]$, respectively.

We need a remark of G. Zappa [11]:

(2.3) *If α is a regular automorphism of a finite group G, then every element g of G can be written in the form $g=x^{-1}x\alpha$ with some $x\in G$.*

Again we repeat that if $(G, \alpha)\in[\mathsf{A}_4]$, then $T(G)$ is the set of all t $(\in G)$ such that $t\alpha^2=t$, and $S(G)$ is the set of all s $(\in G)$ such that $s\alpha^2=s^{-1}$. Clearly $T(G)\alpha=T(G)$, $S(G)\alpha=S(G)$; and $T(G)$ is a subgroup of G.

Finally, two of the results mentioned in the introduction have to be stated precisely for reference in the sequel.

(2.4) (Neumann [8]) *If $(G, \alpha)\in[\mathsf{A}_2\Sigma]$, then G is abelian and $g\alpha=g^{-1}$ for every g in G.*

(2.5) (Nagata [7], Higman [4]) *If G is a locally nilpotent group and if $(G, \alpha)\in[\mathsf{A}_2]$, then G is abelian and $g\alpha=g^{-1}$ for every g in G.*

3. Preparations

We start with a purely technical remark.

(3.1) *If α is a regular automorphism of a finite group G, then an equality of the form $a\alpha = a^g$ $(a, g \in G)$ can hold only for $a = 1$.* Indeed, according to (2.3), $a\alpha = a^g$ yields $a\alpha = a^{x^{-1}x\alpha}$ with some x in G, whence $(a^x)\alpha = a^{x^{-1}}$, and, as α is regular, this can only happen if $a^{x^{-1}} = 1$, $a = 1$.

Next we present four lemmas on A_4- and Σ-groups.

(3.2) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4]$ and G is locally finite, then G is a Σ-group.*

Proof. Take an arbitrary element g of G and consider its α-orbit, that is, the set consisting of the elements g, $g\alpha$, $g\alpha^2$, $g\alpha^3$. The subgroup K generated by these elements is finite and α-invariant; as such, it is the union of a finite number of disjoint α-orbits. Since the α-orbit of any non-identity element of G consists either of two or of four elements, it follows that the number of non-identity elements in K is even, the order of K is odd. Hence g itself is of odd order and, as g was arbitrary, the same holds for every element of G. Thus (B.5.1) proves the lemma.

(3.3) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4\Sigma]$, then $T(G)$ is abelian and $t\alpha = t^{-1}$ for every t in $T(G)$.*

Proof. If $t \in T(G)$ then, using (B.5.3), we see that $(t\sigma)\alpha^2 = (t\alpha^2)\sigma = t\sigma$, so $t\sigma$ also belongs to $T(G)$. This shows that $T(G)$ is a closed subgroup of G; as such, it is itself a Σ-group. If β denotes the restriction of α to $T(G)$, then $(T(G), \beta) \in [\mathsf{A}_2\Sigma]$, and so (2.4) implies the statement of the lemma.

(3.4) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4\Sigma]$, then every element g of G can be written, in one and only one way, as $g = ts$ with $t \in T(G)$, $s \in S(G)$.*

Proof. Let us first assume that $g = ts$, $t \in T(G)$, $s \in S(G)$. Then $g^{-1}\alpha^2 = (s^{-1}t^{-1})\alpha^2 = st^{-1}$, so $g^{-1}\alpha^2 g = s^2$, $s = (g^{-1}\alpha^2 g)\sigma$; $s^{-1} = (s^2)^{-1}\sigma = (g^{-1}g\alpha^2)\sigma$, $t = gs^{-1} = g(g^{-1}g\alpha^2)\sigma$. This shows that no element of G can be written in the form $g = ts$ in more than one way.

On the other hand, using again that $(x\sigma)^{-1} = (x^{-1})\sigma$ with $x = g^{-1}\alpha^2 g$, we always have that $g = g \cdot (g^{-1}g\alpha^2)\sigma \cdot (g^{-1}\alpha^2 g)\sigma$; so we have to prove only that $g(g^{-1}g\alpha^2)\sigma = t \in T(G)$ and $(g^{-1}\alpha^2 g)\sigma = s \in S(G)$ for every g in G. (B.5.3) enables us to argue that

$$\big((g^{-1}\alpha^2 g)\,\sigma\big)\,\alpha^2 = \big((g^{-1}\alpha^2 g)\,\alpha^2\big)\,\sigma = (g^{-1}g\,\alpha^2)\,\sigma = \big((g^{-1}\alpha^2 g)\,\sigma\big)^{-1},$$

so indeed $(g^{-1}\alpha^2 g)\sigma = s \in S(G)$; using this,

$$(g\,s^{-1})\,\alpha^2\,(g\,s^{-1})^{-1} = g\,\alpha^2 s\,s\,g^{-1} = g\,\alpha^2(g^{-1}\alpha^2 g)\,g^{-1} = 1,$$

so that $(gs^{-1})\alpha^2 = gs^{-1}$, $gs^{-1} = t \in T(G)$.

(3.5) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4\Sigma]$, then the subgroup $H(G)$ generated by $S(G)$ contains the derived group G' of G.*

Proof. First one shows that $H(G)$ is normal in G. If $s \in S(G)$ and $t \in T(G)$, then $(s^t)\alpha^2 = (s^{-1})^t = (s^t)^{-1}$, so that $s^t \in S(G)$; whence $S(G)^{T(G)} = S(G)$. As (3.4) gives $G = T(G)\,S(G)$, we have that $S(G)^G = S(G)^{T(G)\,S(G)} = S(G)^{S(G)} \subseteq H(G)$,

and so indeed $H(G)^G = H(G)$. From (3.4) it follows also that $G = T(G)\,H(G)$, whence

$$G/H(G) = T(G)\,H(G)/H(G) \cong T(G)/\big(T(G) \cap H(G)\big),$$

which is abelian according to (3.3). This proves that $H(G)$ contains G'.

On locally nilpotent A_4-groups we have two similar lemmas. The first of these can be proved in the same way as (3.3), except that one has to use (2.5) instead of (2.4).

(3.6) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4]$ and if G is locally nilpotent, then $T(G)$ is abelian and $t\alpha = t^{-1}$ for every t in $T(G)$.*

(3.7) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4]$ and if G is locally nilpotent, then G has no element of order two.*

Proof. Take any element x of G such that $x^2 = 1$, and consider the subgroup K generated by x, $x\alpha$, $x\alpha^2$ and $x\alpha^3$. Then K is generated by a finite number of elements whose orders divide 2, therefore K is nilpotent, finite, and its order is a power of 2. On the other hand, α restricted to K is still a regular automorphism of order dividing 4, so that (3.2) and (B.5.1) imply that the order of K is odd. Hence the order of K can be only 2^0, proving that $x = 1$.

In the following two lemmas we turn to direct preparation for Section 4. There, in course of investigating a certain nilpotent A_4-group, we shall need homomorphic images of lower nilpotency class which admit regular automorphisms induced by the particular automorphism of the original group. Here we prove that such images are provided, for instance, by the homomorphisms corresponding to the closures $\Delta_i(G)$ of the members $\Gamma_i(G)$ of the lower central series of the group; also, that these closures $\Delta_i(G)$ form a central series of the group which inherits an analogue of the well-known property $[\Gamma_i(G), \Gamma_j(G)] \leqq \Gamma_{i+j}(G)$ of the lower central series of the group.

(3.8) Lemma. *If G is a nilpotent group without elements of order two, then $[\Delta_i(G), \Delta_j(G)] \leqq \Delta_{i+j}(G)$ (for every $i, j = 1, 2, \ldots$).*

Proof. This is done by induction on the nilpotency class c of G. For $c = 1$ the statement is obvious. If $c > 1$, we assume that the statement holds whenever the class of the group considered is less than c. — We have $G = Z_c(G)$, $\Gamma_{c+1}(G) = \Delta_{c+1}(G) = \Delta_{c+2}(G) = \cdots = E$, $\Gamma_c(G) \leqq Z_1(G)$. From (B.9.4) we know that $G \in [\mathit{II}]$; by (B.9.2), $Z_1(G)$ is an ideal, so it is a fortiori closed. Hence $\Delta_c(G) = cl\,\big(\Gamma_c(G)\big) \leqq cl\,\big(Z_1(G)\big) = Z_1(G)$, so $[\Gamma_1(G), \Delta_c(G)] = E$; $\Delta_c(G)$ is closed and normal in G, so by (B.9.5) it is an ideal; finally, if $i \leqq c$ then $\Delta_i(G) = cl\,\big(\Gamma_i(G)\big) \geqq cl\big(\Gamma_c(G)\big) = \Delta_c(G)$. Thus (B.10.2) may be applied: $\Delta_i\big(G/\Delta_c(G)\big) = cl\big(\Gamma_i(G/\Delta_c(G))\big) = cl\,\big(\Gamma_i(G)\,\Delta_c(G)/\Delta_c(G)\big) = cl\,\big(\Gamma_i(G)\big)\,\Delta_c(G)/\Delta_c(G) = \Delta_i(G)/\Delta_c(G)$ for $i \leqq c$, and this can be extended to any i by writing it in the form $\Delta_i\big(G/\Delta_c(G)\big) = \Delta_i(G)\,\Delta_c(G)/\Delta_c(G)$. Now $G/\Delta_c(G)$ is of class $c-1$, so the induction hypothesis gives that

$$[\Delta_i\big(G/\Delta_c(G)\big), \Delta_j\big(G/\Delta_c(G)\big)] \leqq \Delta_{i+j}\big(G/\Delta_c(G)\big),$$
$$[\Delta_i(G)\,\Delta_c(G)/\Delta_c(G), \Delta_j(G)\,\Delta_c(G)/\Delta_c(G)] \leqq \Delta_{i+j}(G)\,\Delta_c(G)/\Delta_c(G),$$
$$[\Delta_i(G)\,\Delta_c(G), \Delta_j(G)\,\Delta_c(G)] \leqq \Delta_{i+j}(G)\,\Delta_c(G);$$

whence, using that $\Delta_c(G) \leqq Z_1(G)$, one obtains that $[\Delta_i(G), \Delta_j(G)] \leqq \Delta_{i+j}(G)\Delta_c(G)$. For $i+j \leqq c$ the right hand side is simply $\Delta_{i+j}(G)$ (since $\Delta_{i+j}(G) \geqq \Delta_c(G)$), so for this case the statement is verified. If however $i+j > c$, then $\Delta_{i+j}(G) = E$, and from $[\Gamma_i(G), \Gamma_j(G)] \leqq \Gamma_{i+j}(G) = E$ it follows by (B.9.1) that also $[\Delta_i(G), \Delta_j(G)] = E$. This completes the proof.

(3.9) Lemma. *Assume that* $(G, \alpha) \in [\mathsf{A}_4]$ *and that* G *is nilpotent. Then* α *induces regular automorphisms* α_i *on the* $G/\Delta_{i-1}(G)$, *for every* $i = 1, 2, \ldots$.

Proof. This again is done by induction on the nilpotency class c of G. If $c = 1$, we have nothing to prove: $G/\Delta_{i-1}(G) = G$, $\alpha_i = \alpha$ for all i. If $c > 1$, let us assume that the statement of the lemma is true whenever the class of the group considered is less than c. The cases $i = c$, $c+1, \ldots$ are again obvious.

Take first $G/\Delta_c(G)$; since $\Delta_c(G)$ is characteristic, the mapping α_{c-1} defined by $(g\Delta_c(G))\alpha_{c-1} = (g\alpha)\Delta_c(G)$ is indeed an automorphism of $G/\Delta_c(G)$. If $(g\alpha)\Delta_c(G) = g\Delta_c(G)$, then $g\alpha = gd$ with some $d \in \Delta_c(G)$. (3.7) enables us to recall from the proof of (3.8) that $\Delta_c(G) \leqq Z_1(G)$. Thus $d \in Z_1(G)$, and so $(d\, d\alpha\, d\alpha^2\, d\alpha^3)\alpha = d\, d\alpha\, d\alpha^2\, d\alpha^3$; as α is regular, this means that $d\, d\alpha\, d\alpha^2 d\alpha^3 = 1$. Using this relation and the fact that also $d\alpha$, $d\alpha^2$, $d\alpha^3 \in Z_1(G)$, we get for $h = d^{-1}\alpha^2 \cdot d^{-2}\alpha \cdot d^{-3}$ that

$$h^{-1} h\alpha = d^3\, d^2\alpha\, d\alpha^2\, d^{-1}\alpha^3\, d^{-2}\alpha^2\, d^{-3}\alpha = d^4 (d\, d\alpha\, d\alpha^2\, d\alpha^3)^{-1} = d^4.$$

On the other hand, $g\alpha = gd$ and $d \in Z_1(G)$ yield $g^4\alpha = g^4 d^4$. Thus $d^4 = g^{-4} g^4 \alpha = h^{-1} h\alpha$, therefore $(hg^{-4})\alpha = hg^{-4}$; as α is regular, we have in fact that $g^4 = h \in \Delta_c(G)$. Since $\Delta_c(G)$ is closed, this can only hold if g is in $\Delta_c(G)$. So we have proved that α_{c-1} is regular on $G/\Delta_c(G)$.

For $i < c-1$ we refer to the induction hypothesis. This applies to $G/\Delta_c(G)$ and α_{c-1}: the automorphisms induced by α_{c-1} on the $G/\Delta_c(G)/\Delta_{i+1}(G/\Delta_c(G))$ are regular. Recalling from the proof of (3.8) that $\Delta_{i+1}(G/\Delta_c(G)) = \Delta_{i+1}(G)/\Delta_c(G)$, one sees that these are in fact automorphisms of the $G/\Delta_c(G)/\Delta_{i+1}(G)/\Delta_c(G) \cong G/\Delta_{i+1}(G)$, induced by α through α_{c-1}, and this was to be proved.

4. A class of locally nilpotent A_4-groups

Following our set course, we now proceed with the study of an arbitrary group H which has a regular automorphism α of order four, such that the elements inverted by α^2 generate the whole of H. It seems natural to start by imposing strong conditions on H. For instance, if we assume that H is finite and nilpotent, can we get a bound for its nilpotency class? The answer is in the affirmative, and with the help of the lemmas presented in the previous section the whole argument carries over to the locally nilpotent case. We give it at once in this general form.

(4.0) Theorem. *If* H *is a locally nilpotent group, if* $(H, \alpha) \in [\mathsf{A}_4]$, *and if the elements that are inverted by* α^2 *generate* H, *then* H *is nilpotent of class at most three.*

The proof splits into several steps and occupies the rest of the section, where we always assume that the premisses of the theorem hold.

(4.1.1) The first step is a reduction. Since H is generated by $S(H)$, $\Gamma_4(H)$ is generated by all commutators of the form $[s_1, \ldots, s_4]$ $(s_1, \ldots, s_4 \in S(H))$ together with their conjugates in H. So if we prove that $[s_1, \ldots, s_4] = 1$ for any four elements $s_1, \ldots, s_4$ of $S(H)$, then in fact we prove the theorem. To achieve this, we consider four arbitrary (but for the rest of the proof fixed) elements $s_1, \ldots, s_4$ of $S(H)$, and then the subgroup K generated by $s_1, \ldots, s_4, s_1\alpha, \ldots, s_4\alpha$, and prove that K is nilpotent of class at most three.

(4.1.2) If β denotes the restriction of α to K, then $(K, \beta) \in [\mathsf{A}_4]$. As a finitely generated subgroup of H, K is nilpotent of some class c. Also, $S(K) = S(H) \cap K$, so K is generated by $S(K)$. From (3.7) and (B.9.4) it follows that K is a Π-group.

We want to prove that $c \leqq 3$; it is convenient to restrict ourselves to the case $c \geqq 3$ and then prove that $c = 3$.

(4.1.3) In the rest of the proof, we write briefly Γ_i, Δ_i, and Z_i for $\Gamma_i(K)$, $\Delta_i(K)$, and $Z_i(K)$. From (3.8) it follows that the Δ_i form a central series of K; as the length of such a series cannot be less than c, and as $\Delta_{c-1} = E$, all the $\Delta_1, \Delta_2, \ldots, \Delta_{c+1}$ are distinct. According to (3.9), β induces regular automorphisms β_i on the factor groups $K_i = K/\Delta_{i-1}$ $(i \leqq c)$. Each K_i is nilpotent of class exactly i, and from (B.9.5) it follows that each K_i is a Π-group.

Almost all the steps that follow are concerned with the derivation of congruences between commutators of arbitrary elements of $S(K)$. We use throughout without further mention that congruences modulo Γ_i automatically hold modulo Δ_i, and also that the Δ_i are characteristic in K.

Let then x, y, u, v denote arbitrary elements of $S(K)$; in this case also $x\beta, y\beta, u\beta, v\beta \in S(K)$.

(4.2.1) $[x, y]\beta \equiv [x, y]^{-1} \bmod \Delta_3$.

Proof. Since $[x, y]\beta^2 = [x^{-1}, y^{-1}] \equiv [x, y] \bmod \Delta_3$, we have that $([x, y]\Delta_3)\beta_2^2 = [x, y]\Delta_3$; therefore $[x, y]\Delta_3 \in T(K_2)$. According to (4.1.3), (3.6) applies to $T(K_2)$ and gives that $([x, y]\Delta_3)\beta_2 = ([x, y]\Delta_3)^{-1}$, whence $([x, y]\beta)\Delta_3 = [x, y]^{-1}\Delta_3$ and so (4.2.1) holds.

(4.2.2) $[x, y\beta] \equiv [x\beta, y] \bmod \Delta_3$.

Proof. $[x, y\beta]\beta = [x\beta, y^{-1}] \equiv [x\beta, y]^{-1} \bmod \Delta_3$; on the other hand, by (4.2.1), $[x, y\beta]\beta \equiv [x, y\beta]^{-1}$. Comparing these we get that $[x, y\beta]^{-1} \equiv [x\beta, y]^{-1}$, so indeed $[x, y\beta] \equiv [x\beta, y] \bmod \Delta_3$.

(4.3.1) $[a, b, c] \equiv [a, c, b][b, c, a]^{-1} \bmod \Delta_4$, for every $a, b, c \in K$.

This follows from (2.1).

(4.3.2) $[x, y, u\beta] \equiv [x, u, y\beta][y, u, x\beta]^{-1} \bmod \Delta_4$.

Proof. Apply (4.3.1) with $a=x$, $b=y$, $c=u$, and map the result by β:

$$[[x, y]\beta, u\beta] \equiv [[x, u]\beta, y\beta]\,[[y, u]\beta, x\beta]^{-1} \bmod \Delta_4.$$

Now use (4.2.1) and (3.8):

$$[[x, y]^{-1}, u\beta] \equiv [[x, u]^{-1}, y\beta]\,[[y, u]^{-1}, x\beta]^{-1} \bmod \Delta_4,$$

$$[x, y, u\beta]^{-1} \equiv [x, u, y\beta]^{-1}[y, u, x\beta] \bmod \Delta_4.$$

Taking inverses on both sides, we obtain the statement.

Similar reasoning leads to the following three congruences:

(4.3.3) $[x, y, u\beta] \equiv [x, u\beta, y]\,[y, u\beta, x]^{-1} \bmod \Delta_4.$

(4.3.4) $[x, u, y\beta] \equiv [x, y\beta, u]\,[y, u\beta, x] \bmod \Delta_4.$

(4.3.5) $[y, u, x\beta] \equiv [x, y\beta, u]^{-1}[x, u\beta, y] \bmod \Delta_4.$

(4.3.6) $[x, y, u\beta] \equiv [x, y\beta, u] \bmod \Delta_4.$

Proof. By (4.3.2),

$$[x, y, u\beta]^2 \equiv [x, y, u\beta]\,[x, u, y\beta]\,[y, u, x\beta]^{-1} \bmod \Delta_4.$$

Substituting the terms on the right hand side from (4.3.3), (4.3.4), and (4.3.5), after cancellations one gets that $[x, y, u\beta]^2 = [x, y\beta, u]^2 \bmod \Delta_4$. So

$$([x, y, u\beta]\,\Delta_4)^2 = ([x, y\beta, u]\,\Delta_4)^2$$

and, since $K_3 \in [\Pi]$, this implies that

$$[x, y, u\beta]\,\Delta_4 = [x, y\beta, u]\,\Delta_4.$$

(4.3.7) $[x, y\beta, u\beta]^{-1} \equiv [x, y, u] \bmod \Delta_4.$

This follows from (4.3.6), using (3.8).

(4.4.1) $[x, y, u, v] \equiv [x, y, v, u] \bmod \Delta_5.$

Proof. By (4.2.1), there exist $z, w \in \Delta_3$ such that $[x, y]\beta = [x, y]^{-1}z$ and $[u, v]\beta = [u, v]^{-1}w$. Using (3.8) we obtain that

$$[x, y, [u, v]]\,\beta = [[x, y]^{-1}z, [u, v]^{-1}w] = [[x, y]^{-1}, [u, v]^{-1}] = [x, y, [u, v]] \bmod \Delta_5.$$

Hence $([x, y, [u, v]]\,\Delta_5)\beta_4 = [x, y, [u, v]]\,\Delta_5$, and as β_4 is regular on K_4, this can only hold if $[x, y, [u, v]] \in \Delta_5$. Similarly one verifies that $[u, v^{-1}, [x, y]] \in \Delta_5$. Now apply (2.1) mod Δ_5 with $a=[x, y]$, $b=u$, $c=v$:

$$[x, y, u^{-1}, v]^u\,[u, v^{-1}, [x, y]]^v\,[v, [x, y]^{-1}, u]^{[x, y]} = 1$$

gives that

$$[x, y, u^{-1}, v]\,[v, [y, x], u] \equiv 1 \bmod \Delta_5,$$

whence

$$[x, y, u, v]^{-1}[y, x, v, u]^{-1} \equiv 1, \qquad [x, y, u, v]^{-1}[x, y, v, u] \equiv 1 \bmod \Delta_5.$$

(4.4.2) $[x, y, u, v] \in \Delta_5.$

Proof. $[x, y, u, v]\beta = [[x, y]\beta, u\beta, v\beta]$

$\equiv [x, y, u\beta, v\beta]^{-1} \bmod \Delta_5$, by (4.2.1) and (3.8),

$\equiv [x, y\beta, u, v\beta]^{-1} \bmod \Delta_5$, by (4.3.6) and (3.8),

$\equiv [x, y\beta, v\beta, u]^{-1} \bmod \Delta_5$, by (4.4.1),

$\equiv [x, y, v, u] \bmod \Delta_5$, by (4.3.7) and (3.8),

$\equiv [x, y, u, v] \bmod \Delta_5$, by (4.4.1).

So $([x, y, u, v]\Delta_5)\beta_4 = [x, y, u, v]\Delta_5$; as β_4 is regular on K_4, this can only hold if $[x, y, u, v]\Delta_5 = \Delta_5$.

(4.4.3) *K is nilpotent of class three.*

Proof. By (4.4.2), all the 4-th commutators of the generators $s_1, \ldots, s_4$, $s_1\alpha, \ldots, s_4\alpha$ of K belong to Δ_5; since Δ_5 is normal in K, it contains also their conjugates. Thus Δ_5 contains Γ_4, whence also $\Delta_4 = cl(\Gamma_4) \leqq \Delta_5$, so that in fact $\Delta_4 = \Delta_5$. As the central series $K = \Delta_1 > \cdots > \Delta_{c-1} = E$ was properly descending, this implies that $\Delta_4 = \Delta_5 = E$, a fortiori that $\Gamma_4 = E$, $c \leqq 3$. We assumed that $c \geqq 3$, so in fact $c = 3$.

In view of (4.4.1), this completes the proof of the theorem.

5. A class of (locally) finite and soluble A_4-groups

In the preceding section we started investigating a special class of A_4-groups: a group H belonged to this class if it had a regular automorphism α of order four, such that the elements that were inverted by α^2 generated the whole group H. There we worked under the additional assumption that H was locally nilpotent. Now we pursue the study of *finite* groups of this class, weakening the condition of nilpotency to solubility. Here we have to apply a very finite method: induction on the order of the group. Thus, though the problem is still of local character and admits a generalization in this direction, we cannot, as we did in the preceding section, deal with the generalized case at once.

Let us assume therefore, for the time being, that H is a finite group and that $(H, \alpha) \in [A_4]$. It will be convenient to write simply Γ_i instead of $\Gamma_i(H)$, and when it can be done without danger of ambiguity we shall also use T and S instead of $T(H)$ and $S(H)$. As usual, curly brackets denote (sub)-groups generated by the elements, or by the elements of the sets, listed between them. Let us recall the preparatory results concerning this case:

by (3.2) and (B.5.1), $H \in [\mathfrak{S}]$, and the order of H is odd; so,

by (3.3), T is abelian, and $t\alpha = t^{-1}$ for every t in T;

by (3.4), $H = TS$;

by (2.3), every element h of H can be written in the form $h = x^{-1}x\alpha$ with some x in H;

by (3.1), $h\alpha = h^x$ (for $h, x \in H$) can only hold if $h = 1$.

Then we need four lemmas.

(5.1) Lemma. $[s^t, s\alpha]=[t, s, s\alpha]=1$ *for every* $t\in T$, $s\in S$.

Proof. Let t, s be arbitrary elements of T, S, respectively. Then

$$[s^t, s\alpha]\alpha = [(s\alpha)^{t^{-1}}, s^{-1}] = [s, (s\alpha)^{t^{-1}}]^{s^{-1}} = [s^t, s\alpha]^{t^{-1}s^{-1}},$$

so by (3.1) we have $[s^t, s\alpha]=1$. In particular, $[s, s\alpha]=1$. Thus

$$1=[s^t, s\alpha]=[s[s, t], s\alpha]=[s, s\alpha]^{[s,t]}[s, t, s\alpha]=[[t, s]^{-1}, s\alpha],$$

but then also $[t, s, s\alpha]=1$.

(5.2) Lemma. $[t, s, s, s]=[t, s, s, s\alpha]=1$ *for every* $t\in T$, $s\in S$.

Proof. Take any t from T and any s from S. Define $H_1=\{s, s^t\}$, $H_2=H_1\alpha=\{s\alpha, (s\alpha)^{t^{-1}}\}$, $H_3=\{H_1, H_2\}$, $T_i=H_i\cap T$, $S_i=H_i\cap S$ $(i=1, 2, 3)$. Then $H_i\alpha^2=H_i$ for each i; since the order of H is odd, so is the order of H_i, and then $H_i\in[\Sigma]$. As in (3.4), one proves that $H_i=T_iS_i$. From (5.1) it follows that $[H_1, H_2]=E$, giving that $H_1\cap H_2\leqq Z_1(H_3)$. Since $T_1=T_1\alpha=T_2$, $T_1\leqq H_1\cap H_2\leqq Z_1(H_3)$; in particular, T_1 is normal in H_1. Consider now H_1/T_1. Define $(hT_1)\beta=(h\alpha^2)T_1$ for each h in H_1; since $T_1\alpha^2=T_1$ and $H_1\alpha^2=H_1$, β is an automorphism of H_1/T_1. Assume $(hT_1)\beta=hT_1$, that is, $h\alpha^2=ht$ for some $h\in H_1$, $t\in T_1$. Then $t=h^{-1}h\alpha^2$; but we have seen in the proof of (3.4) that elements of this form belong to S, so $t\in T\cap S=E$. It follows from this that $h\alpha^2=h$, $h\in T_1$, showing that β is regular. But then (3.2) and (2.4) imply that H_1/T_1 is abelian, $H_1'\leqq T_1$. Similarly $H_2'\leqq T_2=T_1$; as we have already seen that $[H_1, H_2]=E$, we get also that $H_3'=\{H_1', H_2'\}\leqq T_1\leqq Z_1(H_3)$. Now

$$[[t, s]^{-1}, s]=[s, t, s]=[s, s]^{[s,t]}[s, t, s]=[s[s, t], s]=[s^t, s]\in H_3'\leqq Z_1(H_3),$$

but then also $[t, s, s]\in Z_1(H_3)$, and our statement follows.

(5.3) Lemma. *Let* N *be any normal subgroup of* H *such that* $T\cap Z_1(N)\neq E$. *If* s *is an arbitrary element of* S, *and if* $N_s=\{N, s, s\alpha\}$, *then* $Z_1(N_s)\neq E$.

Proof. Take any nontrivial element t from $T\cap Z_1(N)$. As $Z_1(N)$ is normal in H, $[t, s, s]\in Z_1(N)$; further, by (5.2), $[t, s, s, s]=[t, s, s, s\alpha]=1$, so in fact $[t, s, s]\in Z_1(N_s)$. If $[t, s, s]=1$, then we consider $[t, s]$; this is also in $Z_1(N)$, and, by (5.1), $[t, s, s\alpha]=1$, so that $[t, s]\in Z_1(N_s)$. Finally, if $[t, s]=1$, then also $1=[t, s]\alpha=[t^{-1}, s\alpha]$, whence $[t, s\alpha]=1$, proving that $t\in Z_1(N_s)$. Therefore one of $[t, s, s]$, $[t, s]$, and t is a nontrivial element in $Z_1(N_s)$.

(5.4) Lemma. *If* N *is any subset of* H *such that* $N^S\subseteqq S$, *then* $[N, S]=E$.

Proof. For any n in N and any s in S, $n^s\in S$, so that $(n^s)\alpha^2=(n^s)^{-1}$; on the other hand, $n=n^1\in S$ implies that $(n^s)\alpha^2=(n^{-1})^{s^{-1}}=(n^{s^{-1}})^{-1}$. So $(n^s)\alpha^2=(n^s)^{-1}=(n^{s^{-1}})^{-1}$, whence $n^s=n^{s^{-1}}$, so that $n^{s^2}=n$, $[n, s^2]=1$. Thus by (B.5.6) also $[n, s]=1$.

Now we are ready to prove the second theorem.

(5.5) Theorem. *Let* H *be a finite soluble group,* $(H, \alpha)\in[\mathsf{A}_4]$, *and assume that the elements that are inverted by* α^2 *generate* H. *Then* H *is nilpotent of class at most three.*

We prove this by induction on the order of H. The unit group provides a starting point, so we only have to give the inductive step. For this we assume that the order of H is greater than 1, and that the theorem holds whenever the order of the group considered is less than the order of H.

Again we split the proof into several steps.

(5.5.1) *If N is a normal subgroup of H such that $N\alpha = N$ and $N \neq E$, then $\Gamma_4(H)$ is contained in N.*

Proof. Define $(hN)\beta = (h\alpha)N$ for every $hN \in H/N$; clearly β is an automorphism of H/N. If $(hN)\beta = hN$, that is, if $h\alpha = hn$ for some h in H and n in N, then $h^{-1}h\alpha = n$. If γ means the restriction of α to N, then $(N, \gamma) \in [\mathsf{A}_4]$. According to (2.3), $n = m^{-1}m\alpha$ with some m in N; but then $n = h^{-1}h\alpha = m^{-1}m\alpha$, whence $(hm^{-1})\alpha = hm^{-1}$; as α is regular, this can only hold if $hm^{-1} = 1$, $h = m \in N$. Thus β is regular on H/N; further, $S(H/N) = S(H)N/N$ generates H/N, so that the inductive hypothesis applies to H/N: $\Gamma_4(H/N) = E$. But $\Gamma_4(H/N) = \Gamma_4(H)N/N$, so in fact $\Gamma_4(H) \leqq N$.

(5.5.2) *If $Z_1(H') = E$, then $H' = E$, so that $\Gamma_4(H) = E$.*

Proof. Suppose that, on the contrary, $H' \neq E$, but $Z_1(H') = E$. Then also $H'' \neq E$. As α can be restricted to H', and $T(H') = T(H) \cap H'$, $S(H') = S(H) \cap H'$, (3.4) gives that $H' = \big(T(H) \cap H'\big)\big(S(H) \cap H'\big)$. Since $T(H) \cap H'$ is abelian but H' is not, $S(H) \cap H' \neq E$. Also H'' is contained in $\{S(H) \cap H'\}$, by (3.5). Consider now $\{S(H) \cap H'\}$. This is a finite soluble group, in fact a proper subgroup of H (being contained in H'). Further, it is an α-invariant subgroup, and so α can be restricted to $\{S(H) \cap H'\}$ to show that it is an A_4-group. Finally, this group is generated by elements that are inverted by α^2. Hence the induction hypothesis implies that $\{S(H) \cap H'\}$ is nilpotent; since H'' is contained in $\{S(H) \cap H'\}$, it is nilpotent too, so that $Z_1(H'') \neq E$. As $Z_1(H'')$ is characteristic in H, (5.5.1) ensures that $\Gamma_4 = \Gamma_4(H)$ is contained in $Z_1(H'')$. On the other hand we know that $H'' \leqq \Gamma_4$, therefore $\Gamma_4 = Z_1(H'') = H'' \neq E$.

Now α can be restricted to Γ_4, so that Γ_4 is an A_4-group and $T(\Gamma_4) = T(H) \cap \Gamma_4$, $S(\Gamma_4) = S(H) \cap \Gamma_4$; therefore, by (3.4), $\Gamma_4 = \big(T(H) \cap \Gamma_4\big)\big(S(H) \cap \Gamma_4\big)$. If $T(H) \cap \Gamma_4 = E$, then $\Gamma_4 = S(H) \cap \Gamma_4 \leqq S(H)$; as Γ_4 is normal in H, $\Gamma_4^{S(H)} = \Gamma_4 \leqq S(H)$ too, so that (5.4) says that $[\Gamma_4, S(H)] = E$. Since $S(H)$ generates the whole of H, this means that $[\Gamma_4, H] = E$; a fortiori $[\Gamma_4, H'] = E$, whence $\Gamma_4 \leqq Z_1(H')$; but $\Gamma_4 \neq E$ while $Z_1(H') = E$, a contradiction. It follows therefore that $T(H) \cap \Gamma_4 \neq E$.

This is the same as saying that $Z_1(H'') \cap T(H) \neq E$. For any s in $S(H) \cap \Gamma_3$, the application of (5.3) with $N = H'' = \Gamma_4$ gives that $Z_1\big(\{\Gamma_4, s, s\alpha\}\big) \neq E$. Now $\{\Gamma_4, s, s\alpha\}$ is normal in H (because $s^h = s[s, h]$, $[s, h] \in \Gamma_4$ for any h in H, etc.), it is also α-invariant, and $Z_1\big(\{\Gamma_4, s, s\alpha\}\big)$ is characteristic in $\{\Gamma_4, s, s\alpha\}$; hence (5.5.1) can be applied to $Z_1\big(\{\Gamma_4, s, s\alpha\}\big)$ to conclude that $Z_1\big(\{\Gamma_4, s, s\alpha\}\big)$ must contain Γ_4. Of this we only need only that $[\Gamma_4, s] = E$ for any $s \in S(H) \cap \Gamma_3$, that is, that $[\Gamma_4, S(H) \cap \Gamma_3] = E$.

Consider next Γ_3. Again it follows that $\Gamma_3 = \big(T(H) \cap \Gamma_3\big)\big(S(H) \cap \Gamma_3\big)$. Since $[T(H) \cap \Gamma_4,\ T(H) \cap \Gamma_3] \leqq T(H)' = E$ and $[T(H) \cap \Gamma_4,\ S(H) \cap \Gamma_3] = E$, we know

that $[T(H)\cap \Gamma_4, \Gamma_3]=E$. This shows that $T(H)\cap Z_1(\Gamma_3)\geqq T(H)\cap\Gamma_4\neq E$. Choose now any s from $S(H)\cap H'$, and apply (5.3) with $N=\Gamma_3$ to obtain that $Z_1(\{\Gamma_3, s, s\alpha\})\neq E$. Again we see that $\{\Gamma_3, s, s\alpha\}$ is normal in H and so on, finally (5.5.1) can be applied to prove that $\Gamma_4\leqq Z_1(\{\Gamma_3, s, s\alpha\})$. This means that $[\Gamma_4, s]=E$ for any s in $S(H)\cap H'$, that is, $[\Gamma_4, S(H)\cap H']=E$. But we have seen that $H'=(T(H)\cap H')(S(H)\cap H')$, so the relations $[T(H)\cap\Gamma_4, T(H)\cap H']\leqq T(H)'=E$ and $[T(H)\cap\Gamma_4, S(H)\cap H']=E$ show that $[T(H)\cap\Gamma_4, H']=E$, $Z_1(H')\geqq T(H)\cap\Gamma_4\neq E$. Thus we have arrived at a statement contradicting the initial hypothesis, and therefore (5.5.2) must hold.

(5.5.3) *If* $Z_1(H')\neq E$, *then* $\Gamma_4(H)=E$.

Proof. As $Z_1(H')$ is characteristic in H, (5.5.1) implies at once that $\Gamma_4\leqq Z_1(H')$. Noting that α can be restricted to $Z_1(H')$, etc., one obtains that $Z_1(H')=(T(H)\cap Z_1(H'))(S(H)\cap Z_1(H'))$. If $T(H)\cap Z_1(H')=E$, then $Z_1(H')=S(H)\cap Z_1(H')\leqq S(H)$; since therefore also $Z_1(H')^{S(H)}=Z_1(H')\leqq S(H)$, (5.4) proves that $[Z_1(H'), S(H)]=E$. As $S(H)$ generates the whole of H, this means that $[Z_1(H'), H]=E$, whence $\Gamma_4\leqq Z_1(H')\leqq Z_1(H)$, $\Gamma_5=E$. — If on the other hand $T(H)\cap Z_1(H')\neq E$, then we can apply (5.3) to $N=H'$ and to any s in $S(H)$; thus $Z_1(\{H', s, s\alpha\})\neq E$. Now $\{H', s, s\alpha\}$ is obviously normal in H, so that also $Z_1(\{H', s, s\alpha\})$ is normal in H; further, it is easy to see that $Z_1(\{H', s, s\alpha\})$ is α-invariant, and then (5.5.1) implies that Γ_4 is contained in $Z_1(\{H', s, s\alpha\})$. Of this we only need that $[\Gamma_4, s]=E$ for every s in $S(H)$; since $S(H)$ generates the whole of H, this means that Γ_4 is in the centre of H, whence $\Gamma_5=E$. — In either case we found that $\Gamma_5=E$, but then (4.0) implies that also $\Gamma_4=E$.

So (5.5.2) and (5.5.3) prove that $\Gamma_4=E$, H is nilpotent of class at most three. This completes the proof of the theorem.

Now let us turn to the infinite case.

(5.6) Theorem. *Let H be a periodic locally soluble group, let $(H, \alpha)\in[\mathsf{A}_4]$, and assume that the elements that are inverted by α^2 generate H. Then H is nilpotent of class at most three.*

Using that a periodic locally soluble group is always locally finite, this theorem can be derived from (5.5) by a routine argument (similar to the one used in the reduction part of the proof of (4.0)).

6. The general case

Let us finally utilize the rather special results of the preceding two sections. Their usefulness is apparent in the light of the fact that if $(G, \alpha)\in[\mathsf{A}_4\mathcal{E}]$ then G' is contained in the subgroup $H(G)$ generated by $S(G)$ and so G' inherits the nilpotency property of $H(G)$. This immediate consequence on G' is further improved in the following theorem.

(6.1) Theorem. *If $(G, \alpha)\in[\mathsf{A}_4\mathcal{E}]$, and if the subgroup $H(G)$ generated by the elements of G which are inverted by α^2 is nilpotent, then the second derived group of G is contained in the centre of G.*

The proof takes several steps. Instead of $H(G)$ we write briefly H.

(6.1.1) $\Gamma_4(H)=E$. This is true because α can be restricted to H, so that (4.0) proves that $\Gamma_4(H)=E$; square roots being unique in H, it follows that also $\Delta_4(H)=E$.

Let Δ be the closure of $\Gamma_3(H)$ in G. As a closed subgroup of the Σ-group G, Δ is itself a Σ-group. H is clearly α-invariant and, according to (3.5), H is normal in G; as $\Gamma_3(H)$ is characteristic in H, it is also normal and α-invariant in G, and then so is Δ.

(6.1.2) $[\Delta, H]=E$.

Proof. Take Δ as the union of the ascending chain of subgroups

$$\Delta^{(1)} \leqq \cdots \leqq \Delta^{(i)} \leqq \cdots$$

defined inductively as follows. Let $\Delta^{(1)}$ be $\Gamma_3(H)$; if $\Delta^{(i)}$ is already defined, then let $\Delta^{(i+1)}$ be the subgroup generated by all x in G such that $x^2 \in \Delta^{(i)}$. Now (6.1.1) implies that $[\Delta^{(1)}, H]=E$, and if $[\Delta^{(i)}, H]=E$, then (B.5.6) implies that also $[\Delta^{(i+1)}, H]=E$. So we have that $[\Delta^{(i)}, H]=E$ for all i, and so in fact $[\Delta, H]=E$.

For the rest of the proof of the theorem, x, y, u, v denote arbitrary elements of $S(G)$, while t and t' are arbitrary elements of $T(G)$.

(6.1.3) $[x, y]\alpha \equiv [x, y]^{-1} \bmod \Delta$.

Proof. Let us work out $[x, y]\alpha^2$:

$$\begin{aligned}
[x, y]\alpha^2 &= [x^{-1}, y^{-1}] = [x, y]^{y^{-1}x^{-1}} = [x, y]\,[x, y, y^{-1}x^{-1}] \\
&= [x, y]\,[x, y, x^{-1}]\,[x, y, y^{-1}]^{x^{-1}} \\
&= [x, y]\,\big([x, y, x]^{-1}\big)^{x^{-1}}\big([x, y, y]^{-1}\big)^{y^{-1}x^{-1}} \\
&= [x, y]\,[x, y, x]^{-1}\,[x, y, y]^{-1} \qquad \text{by (6.1.2)}, \\
&= [x, y]\,\big([x, y, y]\,[x, y, x]\big)^{-1};
\end{aligned}$$

with $d=\big([x, y, y][x, y, x]\big)\sigma \in \Delta$ this can be written as $[x, y]\alpha^2=[x, y]d^2$. From the proof of (3.4) we know that $(g^{-1}g\alpha^2)\sigma \in S(G)$, $g(g^{-1}g\alpha^2)\sigma \in T(G)$ for every g in G. Applying this to $g=[x, y]$, we get that $d^{-1} \in S(G)$ and that $[x, y]d^{-1} \in T(G)$. Now (3.3) yields that $\big([x, y]d^{-1}\big)\alpha=\big([x, y]d^{-1}\big)^{-1}$, whence $[x, y]\alpha=d[x, y]^{-1}d\alpha$; using (6.1.2) we see that this is the same as $[x, y]\alpha=[x, y]^{-1}dd\alpha$, so that $[x, y]\alpha \equiv [x, y]^{-1} \bmod \Delta$.

(6.1.4) $[x, y\alpha] \equiv [x\alpha, y] \bmod \Delta$.

Proof. Since $y\alpha \in S(G)$, we may substitute $y\alpha$ for y in (6.1.3):

$$[x, y\alpha]\alpha \equiv [x, y\alpha]^{-1} \bmod \Delta.$$

But also

$$[x, y\alpha]\alpha = [x\alpha, y^{-1}] = \big([x\alpha, y^{-1}]\big)^{y^{-1}} = [x\alpha, y^{-1}][[x\alpha, y^{-1}], y^{-1}] \equiv [x\alpha, y]^{-1} \bmod \Delta,$$

so we have that $[x, y\alpha]^{-1} \equiv [x\alpha, y]^{-1} \bmod \Delta$. Taking inverses on both sides, we get the required congruence.

(6.1.5) $[a, b, c] = [a, c, b][c, b, a]$ *for every* $a, b, c \in H$.

This follows from (2.1) and (6.1.1).

Now one can work out relations similar to (4.3.2)–(4.3.7) using an exact analogue of the corresponding part of Section 4, consistently referring

to Δ	instead of Δ_3,
E	Δ_4,
equality	congruence mod Δ_4,
(6.1.2)	$[\Delta_3, K] \leqq \Delta_4$,
(6.1.3)	(4.2.1),
(6.1.4)	(4.2.2),
(6.1.5)	(4.3.1).

The relations so obtained, in particular those corresponding to (4.3.6) and (4.3.7), with further application of (6.1.2) and (6.1.4), yield the following line of equalities:

(6.1.6) $[x, y, u]\alpha = [x\alpha, y, u]^{-1} = [x, y\alpha, u]^{-1} = [x, y, u\alpha]^{-1}$.

(6.1.7) $[x^t, y^t, u] = [x, y, u]$.

Proof. While proving (6.1.3) we saw that, for a certain d in Δ, $[x, y]d^{-1} \in T(G)$. By (3.3), $T(G)$ is abelian and so $([x, y]d^{-1})^t = [x, y]d^{-1}$; whence $[x, y]^t = [x, y]d^{-1}d^t$. Put $d^{-1}d^t = d'$; as Δ is normal in G, this element belongs to Δ. Then, by (6.1.2),

$$[x^t, y^t, u] = [[x, y]^t, u] = [[x, y]d', u] = [x, y, u]^{d'}[d', u] = [x, y, u].$$

(6.1.8) $[x, y, u^t] = [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}]$.

Proof. As $S(G)^{T(G)} = S(G)$ (see the proof of (3.5)), x^t, $x^{t^{-1}}$, $y^{t^{-1}}$, u^t all belong to $S(G)$. This makes it possible to apply the relations proved above:

$$\begin{aligned}
[x, y, u^t] &= [x, u^t, y][u^t, y, x], && \text{by (6.1.5)},\\
&= [x^{t^{-1}}, u, y][u, y^{t^{-1}}, x], && \text{by (6.1.7)},\\
&= [x^{t^{-1}}, u, y][u, x, y^{t^{-1}}][x, y^{t^{-1}}, u], && \text{by (6.1.5)},\\
&= [x^{t^{-1}}, u, y][u, x, y^{t^{-1}}][x^t, y, u], && \text{by (6.1.7)},\\
&= [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}], && \text{by (6.1.1)}.
\end{aligned}$$

(6.1.9) $[x, y, u, t] = 1$.

Proof. Let us write down (6.1.8) with $u\alpha^{-1}$ $(\in S(G))$ in the place of u; mapping both sides by α, changing to inverses and using (6.1.6) we obtain that $[x, y, u^{t^{-1}}] = [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}]$. Comparing this with the original form of (6.1.8) we get $[x, y, u^{t^{-1}}] = [x, y, u^t]$. Transform this equality by t and use (6.1.7) to obtain $[x, y, u] = [x, y, u^{t^2}]$. From (6.1.7) it follows

also that $[x, y, u]^{t^2} = [x, y, u^{t^2}]$. Comparison of the last two equalities shows that $[x, y, u]^{t^2} = [x, y, u]$, so that $[x, y, u, t^2] = 1$. Then (B.5.6) proves that also $[x, y, u, t] = 1$.

(6.1.10) $[\Delta, G] = E$.

Proof. As H is generated by $S(G)$, $\Gamma_3(H)$ is the normal closure (in H) of the subgroup generated by all commutators of the form $[x, y, u]$. From (6.1.2) and (6.1.9) we know that $[x, y, u]$ commutes with both of $S(G)$ and $T(G)$; as these two sets generate G, $[x, y, u]$ commutes with the whole of G. Consequently, the subgroup generated by the $[x, y, u]$ is in the centre of G, thus it is its own normal closure $\Gamma_3(H)$, whence $[\Gamma_3(H), G] = E$. Again (B.5.6) is needed to conclude that $[\Delta, G] = E$.

(6.1.11) $[x, y, t] = 1$.

Proof. In the course of proving (6.1.3) we saw that, with a suitable d ($\in \Delta$), $[x, y] d^{-1} \in T(G)$. Since by (3.3) $T(G)$ is abelian, $[[x, y] d^{-1}, t] = 1$. But $[[x, y] d^{-1}, t] = [x, y, t]^{d^{-1}} [d^{-1}, t] = [x, y, t]$ by (6.1.10), and so $[x, y, t] = 1$.

(6.1.12) $[x, y, [u, t]] = 1$.

Proof.
$$\begin{aligned}[x, y, [u, t]] &= [x, y, u^{-1} t^{-1} u t] \\ &= [x, y, t] [x, y, u]^t [x, y, t^{-1}]^{ut} [x, y, u^{-1}]^{t^{-1} ut} \\ &= [x, y, u]^t [x, y, u^{-1}]^{ut}, \quad \text{by (6.1.9) and (6.1.11),} \\ &= [x, y, u]^t \left([x, y, u]^{-1}\right)^{u^{-1} ut} = 1.\end{aligned}$$

(6.1.13) $[u, t, [v, t'], T(G)] = E$.

Proof. Using (6.1.1), $[u, t, [v, t']] = [u^{-1} u^t, v^{-1} v^{t'}]$
$$\begin{aligned}&= [u^{-1}, v^{t'}]^{u^t} [u^t, v^{t'}] [u^{-1}, v^{-1}]^{u^t v^{t'}} [u^t, v^{-1}]^{v^{t'}} \\ &= [u^{-1}, v^{t'}] [u^{-1}, v^{t'} u^t] [u^t, v^{t'}] [u^{-1}, v^{-1}] [u^{-1}, v^{-1}, v^{t'}] [u^{-1}, v^{-1}, u^t] \times \\ &\quad \times [u^t, v^{-1}] [u^t, v^{-1}, v^{t'}];\end{aligned}$$

as $u^t, v^{t'} \in S(G)$, this shows that $[u, t, [v, t']]$ is a product of commutators of the form $[x, y]$ and $[x, y, u]$. Since by (6.1.11) and (6.1.9) these all commute with $T(G)$, so does $[u, t, [v, t']]$.

(6.1.14) $[u, t, [v, t'], S(G)] = E$.

Proof. Take any x from $S(G)$ and expound $[u, t, [v, t'], x] = [u^{-1} u^t, v^{-1} v^{t'}, x]$ using (6.1.1); it becomes

$$[u, v, x] [u, v^{t'}, x]^{-1} [u^t, v, x]^{-1} [u^t, v^{t'}, x].$$

Now if one splits up each term according to (6.1.5), uses (6.1.7) and (6.1.9), then after cancellations one is left with 1.

(6.1.15) *G' is generated by the commutators of the form $[u, v]$ together with those of the form $[u, t]$.*

Proof. (3.4) can be modified to show that any two elements g_1, g_2 of G can be written as $g_1 = ut$, $g_2 = vt'$ with some $u, v \in S(G)$, $t, t' \in T(G)$. Then

$$\begin{aligned}[g_1, g_2] &= [ut, vt'] = [u, t']^t [t, t'] [u, v]^{tt'} [t, v]^{t'} \\ &= [u^t, t'] [u, v]^{tt'} [t, v^{t'}], \qquad \text{(since ,by (3.3), } T(G)' = E\text{)}, \\ &= [u^t, t'] [u, v] [v^{t'}, t]^{-1}, \qquad \text{by (6.1.11)};\end{aligned}$$

as u^t and $v^{t'}$ belong to $S(G)$, one can see that $[g_1, g_2]$ is a product of commutators of the required form. Since every element of G' is a product of commutators like $[g_1, g_2]$, this proves the statement.

(6.1.16) $G'' \leqq Z_1(G)$.

Proof. Let us consider the subgroup K generated by the (ordinary second) commutators formed from the generators of G' which are given in (6.1.15). Every $[x, y, [u, v]]$ is trivial by (6.1.1), and every $[x, y, [u, t]]$ is trivial by (6.1.12), so that in fact K is generated by the $[u, t, [v, t']]$. On the other hand, according to (6.1.13) and (6.1.14), all the $[u, t, [v, t']]$ commute with $T(G)$ and $S(G)$. Since these two sets generate G (see (3.4)), this means that K is central in G and a fortiori normal in G'. But the normal closure of K in G' is just G'', and so we have that $G'' = K \leqq Z_1(G)$.

Thus we arrive at the main results of the paper.

(6.2) Theorem. *If G is a locally nilpotent group, and if $(G, \alpha) \in [\mathsf{A}_4]$, then the second derived group of G is contained in the centre of G.*

Proof. According to (3.7), we can apply (2.3). This says that there exists a locally nilpotent Σ-group G^* such that (i) G^* contains G, (ii) to every element g^* of G^* there is a number $n = n(g^*)$ for which $(g^*)^{2^n} \in G$, and (iii) α can be extended in a unique way to an automorphism β of G^*. It is easy to see that β is regular: if $g^*\beta = g^*$, then the equalities $(g^*\beta)^{2^n} = \big((g^*)^{2^n}\big)\beta = \big((g^*)^{2^n}\big)\alpha = (g^*)^{2^n}$ show that, since α itself is regular, $(g^*)^{2^n} = 1$; as G^* is a Σ-group, this means that $g^* = 1$. Further, we verify that the order of β divides 4: the relations $(g^*\beta^4)^{2^n} = \big((g^*)^{2^n}\big)\beta^4 = \big((g^*)^{2^n}\big)\alpha^4 = (g^*)^{2^n}$ give that $g^*\beta^4 = g^*$, for every $g^* \in G^*$. Finally, β can be restricted to $H(G^*)$ and then (4.0) proves that $H(G^*)$ is nilpotent. So we can apply (6.1) to G^*, and the conclusion is obviously inherited by G.

(6.3) Theorem. *If $(G, \alpha) \in [\mathsf{A}_4]$, and if G is periodic and locally soluble, then the second derived group of G is contained in the centre of G.*

Proof. As a periodic locally soluble group, G is locally finite, so that (3.2) shows that G is a Σ-group. Clearly $H(G)$ is also periodic and locally soluble, and α can be restricted to $H(G)$, so that (5.6) implies that $H(G)$ is nilpotent. Thus we can refer to (6.1), and the proof is complete.

7. Examples

In this last section we give some examples to show that the results presented in this paper are in a sense the best possible. We confine ourselves to the description of the examples, omitting the straightforward but sometimes tedious verifications.

(7.1) First we give the smallest non-metabelian A_4-group. Let us start with a free nilpotent-of-class-two group K_0 on the five generators $s_0, s_1, \ldots, s_4$. Imposing exponent three and the relations $[s_i, s_j][s_{j+k}, s_{i-k}] = 1$ $(i, j, k = 0, \ldots, 4;\ j+k$ and $i+k$ to be taken mod 5), we get a factor group K_1 of K_0. It can be seen that K_1' is generated by $[s_0, s_1]$ and $[s_0, s_2]$, and that these commutators are independent, so that the order of K_1 is 3^7. Now take an isomorphic copy K_2 of K_1 generated in a corresponding way by $r_0, r_1, \ldots, r_4$. Let H be the generalized direct product of K_1 and K_2 amalgamating $[s_0, s_1]$ with $[r_0, r_1]^{-1}$ and $[s_0, s_2]$ with $[r_0, r_2]^{-1}$. Then H is of order 3^{12}, and it can be seen that the mapping τ defined by $s_i\tau = s_{i-1}$, $r_i\tau = r_{i-1}$ is an automorphism of H of order 5. Finally, let us take the splitting extension G of H by an element t corresponding to τ, such that $t^5 = 1$, $s_i^t = s_{i-1}$ and $r_i^t = r_{i-1}$. Clearly G is of order $5 \cdot 3^{12}$; further, it is soluble and has derived length three, but it is not nilpotent. A regular automorphism α of order four can be defined on G by $s_i\alpha = r_i$, $r_i\alpha = s_i^{-1}$, and $t\alpha = t^{-1}$.

(7.2) Repeating exactly the above construction but replacing 5 by some power of 3, one can get non-metabelian A_4-groups which are nilpotent of arbitrarily high class. The direct product of an infinity of such groups of increasing class provides a locally nilpotent but not nilpotent A_4-group.

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Ein komplexer Neokörper ohne reellen Unterneokörper

Von
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Unter einem *komplexen Neokörper* versteht man eine topologische Doppelloop $(K, +, \cdot)$, deren multiplikative Struktur $(K, \cdot)$ der topologischen Halbgruppe aller komplexen Zahlen mit der Multiplikation als Verknüpfung iseomorph ist und in der das Rechtsdistributivgesetz $(x+y)z = xz + yz$ und wegen der Kommutativität der Multiplikation dann natürlich auch das Linksdistributivgesetz $z(x+y) = zx + zy$ gelten; die additive Struktur ist eine topologische Loop mit 0 als neutralem Element (s. z.B. [1], S. 37, 45, 51). Analog ist ein *reeller Neokörper* definiert (loc. cit.). Es ist bekannt, daß eine topologische Doppelloop, die eine topologische 2-Mannigfaltigkeit ist und in der das Rechtsdistributivgesetz gilt, einem komplexen Neokörper iseomorph ist oder einen reellen Neokörper (als Zentrum) enthält; gelten sogar beide Distributivgesetze und erzeugt jedes Element eine multiplikative Gruppe, so ist K sicher ein komplexer Neokörper ([1], S. 54). Es ist hingegen nicht richtig, daß ein komplexer Neokörper stets einen reellen Neokörper enthalten müsse; dies ist insofern bemerkenswert, als ein vierdimensionaler Neokörper (die Dimension eines lokalkompakten zusammenhängenden Neokörpers ist 1, 2 oder 4) allemal einen eindimensionalen Neokörper enthält, der das volle Zentrum darstellt. Ich habe in [1] ein Beispiel einer topologisch algebraischen Struktur angegeben, in der auf dem Körper K der komplexen Zahlen eine neue „Addition" $(x, y) \to px + y$ mit $p \neq 0, 1$ definiert ist, und die keine eindimensionale Unterstruktur bezüglich Multiplikation und neuer Addition enthält. Die in [1], S. 67 aufgestellte Behauptung, $(K, \oplus, \cdot)$ mit $x \oplus y = px + y$ sei ein komplexer Neokörper, ist jedoch falsch, da $(K, \oplus)$ keine topologische Loop, sondern nur eine topologische Quasigruppe mit 0 als Linksneutralelement bildet. Mit der folgenden kurzen Bemerkung soll die offen gebliebene Lücke ausgefüllt werden; wir geben ein Beispiel eines komplexen Neokörpers mit kommutativer Addition an, der keinen reellen Unterneokörper enthält. Die Methoden ähneln in ihrer Verwendung elementarer reeller Analysis dem in [2], S. 229 angewandten Verfahren, das zu einem komplexen Neokörper mit kommutativer und nicht assoziativer Addition führte, in der dennoch jedes Element eine additive Gruppe erzeugt.

Satz. *Ist $(K, +, \cdot)$ der Körper der komplexen Zahlen und ist die Verknüpfung $\oplus$ definiert durch*

$$(0) \qquad x \oplus y = \begin{cases} (x+y) \cdot \exp\left(\frac{34\pi i}{k} \cdot \frac{|xy|^2}{|x+y|^4 + |xy|^2}\right) & \text{für} \quad (x, y) \neq (0, 0) \\ 0 & \text{für} \quad (x, y) = (0, 0) \end{cases}$$

mit einer hinreichend großen natürlichen Zahl k, so ist $(K, \oplus, \cdot)$ ein komplexer Neokörper mit kommutativer Addition ohne einen eindimensionalen Unterneokörper; vielmehr ist der von 1 algebraisch erzeugte Unterneokörper dicht in K.

(Die genaue Gestalt der Konstanten $34\pi/k$ in (0) ist natürlich unwesentlich; wählen wir sie so, so ist $1 \oplus 1 = 2e^{2\pi i/k}$, und die k-fache Iterierte der Funktion $f\colon x \to x \oplus x$ liefert für $x=1$ den reellen Wert 2^k.)

Beweis. Zur Abkürzung bezeichnen wir die reellwertige Funktion $(x, y) \to \frac{34\pi}{k} \cdot |xy|^2 \left(|x+y|^4 + |xy|^2\right)^{-1}$ mit c; sie ist definiert auf $K \times K \setminus \{(0,0)\}$.

a) Aus $c(x, y) = c(y, x)$ und $c(xz, yz) = c(x, y)$ für alle von $(0, 0)$ verschiedenen Paare (xz, yz), $z \in K$ folgt sofort die Kommutativität der Addition und die Gültigkeit der beiden Distributivgesetze.

b) Wegen $|x+y| = |x \oplus y|$ ist die Addition auch im Punkt $(0, 0)$ stetig.

c) Wir zeigen die Existenz einer stetigen Abbildung $(y, z) \to X(y, z)$ von $K \times K$ auf K, so daß die Gleichung $x \oplus y = z$ stets mit $x = X(y, z)$ gleichwertig ist.

Als notwendige Bedingung für die Lösbarkeit der Gleichung $x \oplus y = z$ nach x bekommen wir für $(x, y) \neq (0, 0)$ die Beziehung

$$x = -y + z\, e^{-ic(x,y)}. \tag{1}$$

Setzen wir $y = ue^{ai}$ und $z = ve^{-bi}$ mit $u, v \geqq 0$, $u+v > 0$, so ist dabei

$$c(x, y) = \frac{34\pi}{k} \cdot \frac{u^2|x|^2}{v^4 + u^2|x|^2}, \tag{2}$$

da $|x+y| = |x \oplus y| = |z| = v$ gilt. Der Betrag von x wird ermittelt aus

$$\begin{cases} |x|^2 = |u\, e^{ai} - v\, e^{-(b+c)i}|^2 = u^2 + v^2 - 2uv\cos(a+b+c) \\ \qquad\qquad = (u-v)^2 + 4uv\sin^2\dfrac{a+b+c}{2}. \end{cases} \tag{3}$$

Dabei steht c als Abkürzung für $c(x, y)$.

(2) und (3) ergeben zusammen als notwendige Bedingung für die Lösbarkeit der Gleichung $x \oplus y = z$ bei $(y, z) \neq (0, 0)$ die Existenz einer nichtnegativen reellen Zahl c, die der folgenden Gleichung genügt:

$$\frac{k}{34\pi} \cdot c = \frac{u^4 + u^2v^2 + 2u^3v\cos(a+b+c)}{u^4 + v^4 + u^2v^2 - 2u^3v\cos(a+b+c)}. \tag{4}$$

Die rechte Seite dieser Gleichung setzen wir gleich $f(c; y, z)$ und differenzieren diesen Ausdruck partiell nach c:

$$\begin{cases} f_c(c; y, z) = \dfrac{2u^3v^5\sin(a+b+c)}{(u^4 + v^4 + u^2v^2 - 2u^3v\cos(a+b+c))^2} \\ \qquad = \dfrac{2u^3v^5\sin(a+b+c)}{\left(v^4 + u^2\left((u-v)^2 + 4uv\sin^2\dfrac{a+b+c}{2}\right)\right)^2}. \end{cases} \tag{5}$$

Aus dieser letzten Gleichung bekommen wir die Abschätzungen

$$|f_c(c; y, w)| \leqq \frac{2u^3 v^5}{v^{4\cdot 2}} = \frac{2u^3}{v^3} \quad \text{für} \quad v \neq 0 \tag{6}$$

und

$$|f_c(c; y, w)| \leqq \frac{2u^3 v^5}{(u^2(u-v))^2} = \frac{2v^5}{u(u-v)^4} \quad \text{für} \quad u \neq 0,\ v \neq u \tag{7}$$

Setzen wir für den Augenblick $h = \frac{u}{v}$, wenn $v \neq 0$, und $h = \infty$, wenn $v = 0$, $u \neq 0$, so gilt $\min\left(u^3/v^3,\ v^5/u(u-v)^4\right) = \min\left(h^3,\ 1/h(1-h)^4\right)$, und dieses Minimum ist nicht größer als eine gewisse, feste positive reelle Zahl A. Demnach ist $|f_c(c; y, z)| \leqq 2A$, und dies hat sicher

$$|f_c(c; y, z)| \leqq \frac{k}{68\pi} \tag{8}$$

zur Folge, wenn k von vorneherein nicht kleiner als $136A\pi$ gewählt wird. Aus (8) folgt aber jetzt in der Tat, daß (4) eine und nur eine Lösung $c = \bar{c}(y, z)$ hat, wenn nur $(u, v) \neq (0, 0)$, d.h. $(y, z) \neq (0, 0)$ ist, denn $c \to \frac{k}{34\pi} \cdot c - f(c; y, z)$ ist dann eine streng monotone Funktion, die überdies analytisch ist und auch von u, v, a und b (reell) analytisch abhängt. Aus dieser letzten Bemerkung folgt, daß in dem angegebenen Bereich $\{(y, z) : (u, v) \neq (0, 0)\}$ die Funktion $\bar{c}$ stetig ist. Wir setzen nun

$$X(y, z) = \begin{cases} -y + z \exp -i\bar{c}(y, z) & \text{für} \quad y \neq 0 \quad \text{oder} \quad z \neq 0 \\ 0 & \text{für} \quad y = z = 0. \end{cases} \tag{9}$$

X ist überall stetig.

Für $(y, z) \neq (0, 0)$ bekommen wir

$$X(y, z) \oplus y = z \exp i\left(c(x(y, z), y) - \bar{c}(y, z)\right).$$

Nun ist aber

$$c\left(X(y, z), y\right) = \frac{34\pi}{k} \cdot \frac{|X(y, z)|^2 |y|^2}{|X(y, z) + y|^4 + |X(y, z)|^2 |y|^2},$$

und wegen $|X(y, z) + y| = |X(y, z) \oplus y| = |z|$ ist dieser Ausdruck mit $y = ue^{ai}$ und $z = ve^{-bi}$ gerade gleich

$$\frac{34\pi}{k} \cdot \frac{u^4 + u^2 v^2 - 2u^3 v \cos(a + b + \bar{c}(y, z))}{u^4 + v^4 + u^2 v^2 - 2u^3 v \cos(a + b + \bar{c}(y, z))}.$$

Nach der Definition von $\bar{c}(y, z)$ ist dieser Quotient jedoch genau gleich $\bar{c}(y, z)$; daraus folgt also $c\left(X(y, z), y\right) = \bar{c}(y, z)$ und daraus schließlich $X(y, z) \oplus y = z$. Im Fall $(y, z) = (0, 0)$ ist diese Beziehung nach (9) trivialerweise erfüllt. Der Beweis hat im Fall $(y, z) \neq (0, 0)$ die Eindeutigkeit der Lösung der Gleichung $x \oplus y = z$ nach x gleich mitgeliefert; ist aber $y = z = 0$, so ist wegen $x \oplus 0 = x$ auch in diesem Fall die Lösung eindeutig bestimmt.

Damit haben wir bewiesen, daß $(K, \oplus)$ eine kommutative topologische Loop ist. Demnach ist $(K, \oplus, \cdot)$ ein komplexer Neokörper mit kommutativer Addition. Man beachte, daß die Addition von $(K, \oplus, \cdot)$ außer in $(0, 0)$ sogar analytisch ist.

d) Enthielte $(K, \oplus, \cdot)$ einen eindimensionalen (und dann reellen) Unterneokörper $(H, \oplus, \cdot)$, so wäre $(H \setminus \{0\}, \cdot)$ eine eindimensionale Untergruppe von $(K \setminus \{0\}, \cdot)$, die im Punkt 1 des zweidimensionalen Vektorraums $(K, +)$ eine wohlbestimmte Tangentenrichtung hat; ist h ein reelles Element aus H mit $0 < h < 1$ — man beachte, daß solche Elemente stets existieren —, so liefert $\lim_{n\to\infty} h^{-n}\left((1 \oplus h^n) - 1\right)$ einen Tangentialvektor an H in 1, falls dieser Limes existiert und von 0 verschieden ist. Nun ist zunächst

$$c(1, h^{-n}) = 34\pi/k \cdot h^{-2n}\left((1 + h^{-n})^4 + h^{-2n}\right)^{-1} = 34\pi/k \cdot h^{2n}\left(1 + h^n O_1(h^n)\right)^{-1}$$
$$= h^{2n} O_2(h^n)$$

mit $$\overline{\lim_{n\to\infty}} |O_i(h^n)| < \infty, \quad i = 1, 2.$$

Damit wird

$$h^{-n}\left((1 \oplus h^n) - 1\right) = h^{-n}\left\{(1 + h^n) \exp i\, c(1, h^{-n}) - 1\right)$$
$$= \exp i\, c(1, h^{-n}) + h^{-n}\left(i\, h^{2n} O_2(h^n) + \cdots\right) = 1 + h^n O_3(h^n).$$

Demnach existiert der Grenzwert von $h^{-n}\left((1 \oplus h^n) - 1\right)$ und ist gleich 1. Die einzige eindimensionale Untergruppe $(H \setminus \{0\}, \cdot)$ von $(K \setminus \{0\}, \cdot)$, die in 1 den Tangentialvektor 1 hat, ist aber die Untergruppe aller reellen Zahlen von $K\{0\}$. Nun ist aber schon $1 \oplus 1 = 2e^{2\pi i/k}$ nicht reell, also kann das algebraische Erzeugnis von 1 nicht in H enthalten sein und ist daher in K dicht [1], S. 50. Wir geben jetzt noch das Element h explizit an: Es genügt $h^{-1} = 2^k$, also $h = 2^{-k}$ zu nehmen; bezeichnen wir nämlich mit f die Funktion $x \to x \oplus x$, so gilt mit $f_n = f \circ f_{n-1}$ $(n = 1, \ldots, f_0 =$ identische Abbildung) die Beziehung $f_n(1) = 2^n \exp 2\pi i n/k$; demnach liegt also das Element $h^{-1} = f_k(1)$ tatsächlich in der von 1 erzeugten additiven Loop und h in dem von 1 erzeugten Neokörper.

Damit ist der Beweis des Satzes vollendet.

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On Brauer's Theorem on Sections in the Theory of Blocks of Group Characters

By

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R. Brauer, in his earlier paper [3], announced an important theorem concerning both "blocks" of group characters and "sections" in groups. In his recent paper [5], he gave a proof of the theorem, in which certain idempotents of the centers of group rings are used together with linear functions defined on the centers. In this note, we shall give another proof of the theorem by making use of the idempotents (§1) and, as a consequence of this, have some results on the idempotents (§2).

Let $\mathfrak{G}$ be a group of finite order g and p a fixed rational prime. The terms "modular character" and "block" without modifier will always mean modular character and block for p. Let $\chi_1, \chi_2, \ldots, \chi_n$ be the irreducible ordinary characters of $\mathfrak{G}$. For an arbitrarily fixed p-element P of $\mathfrak{G}$, we shall consider the normalizer $\tilde{\mathfrak{G}} = \mathfrak{N}(P)$ of P in $\mathfrak{G}$. The generalized decomposition numbers $d_{i\varkappa}^P$ of $\mathfrak{G}$ belonging to P are defined by

$$\chi_i(PR) = \sum_{\varkappa=1}^{r} d_{i\varkappa}^P \varphi_\varkappa^P(R) \tag{1}$$

([1]), where $\varphi_1^P, \varphi_2^P, \ldots, \varphi_r^P$ are the irreducible modular characters of $\tilde{\mathfrak{G}}$ and R is a variable p-regular element of $\tilde{\mathfrak{G}}$. The above-mentioned theorem of R. Brauer is described as follows:

Theorem 1. (Brauer) *If a modular character $\varphi_\varkappa^P$ of $\tilde{\mathfrak{G}}$ belongs to a block $\tilde{B}_\sigma$ of $\tilde{\mathfrak{G}}$, then $d_{i\varkappa}^P$ can be different from zero only for ordinary characters χ_i of $\mathfrak{G}$ which belong to the block B_τ of $\mathfrak{G}$ determined by $\tilde{B}_\sigma$ i.e. $B_\tau = \tilde{B}_\sigma^{\mathfrak{G}}$* ([3], [5]).

Let $\tilde{B}^{(\tau)}$ be the collection of blocks $\tilde{B}_\sigma$ of $\tilde{\mathfrak{G}}$ which determine an arbitrarily given block B_τ of $\mathfrak{G}$. For each non-vacuous $\tilde{B}^{(\tau)}$ we shall consider the matrix

$$D_0 = (d_{i\varkappa}^P), \tag{2}$$

where the row index i ranges over those values for which $\chi_i \in B_\tau$ and the column index $\varkappa$ ranges over those values for which $\varphi_\varkappa^P \in \tilde{B}^{(\tau)}$. In the proof

of the theorem carried out in [5], the main point is to establish the inequality

$$D_0' \overline{D}_0 \neq 0, \tag{3}$$

where D_0' is the transposed matrix of D_0 and $\overline{D}_0$ is the matrix whose coefficients $\overline{d}_{i\varkappa}^P$ are complex conjugate to $d_{i\varkappa}^P$.

Let $\mathfrak{p}$ be a fixed prime ideal divisor of p in the field Ω of g-th roots of unity and $\mathfrak{o}_\mathfrak{p}$ be the ring of $\mathfrak{p}$-integers in Ω. Another essential part of the proof is to verify that the coefficients of the matrix $C_0^{-1} D_0' \overline{D}_0$ are all in $\mathfrak{o}_\mathfrak{p}$, where C_0 is the submatrix of the Cartan matrix C_P of $\tilde{\mathfrak{G}}$ associated with $\tilde{B}^{(\tau)}$.

In this note, we shall establish these two facts by making use of a type of expressions related with the idempotents (see (6)). In this case, (3) will be obtained more sharply in the form

$$C_0^{-1} D_0' \overline{D}_0 \equiv E \pmod{\mathfrak{p}} \tag{4}$$

where E is the unit matrix whose degree is equal to the number of modular characters $\varphi_\varkappa^P$ belonging to $\tilde{B}^{(\tau)}$.

1. The proof of BRAUER's theorem. Let $K_1, K_2, \ldots, K_n$ be the classes of conjugate elements in the group $\mathfrak{G}$ and $\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_m$ be those of $\tilde{\mathfrak{G}} = \mathfrak{N}(P)$. We denote by $\mathfrak{S}(P)$ the section of P in $\mathfrak{G}$, i.e. the set of all elements whose p-factors are conjugate to P in $\mathfrak{G}$. Assuming that $\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_r$ are the p-regular classes of $\tilde{\mathfrak{G}}$, $P\tilde{K}_1, P\tilde{K}_2, \ldots, P\tilde{K}_r$ are the classes of $\tilde{\mathfrak{G}}$ which are contained in the section $\tilde{\mathfrak{S}}(P)$ of P in $\tilde{\mathfrak{G}}$. As is well known, arranging $K_1, K_2, \ldots, K_n$ in a suitable order, we may assume that each $P\tilde{K}_\alpha$ is contained in K_α, $\alpha = 1, 2, \ldots, r$. In this case, $K_1, K_2, \ldots, K_r$ are the classes of $\mathfrak{G}$ contained in $\mathfrak{S}(P)$.

Let e_i be the primitive idempotent of the center Z of the group ring of $\mathfrak{G}$ over Ω which belongs to χ_i. After M. OSIMA [8], we shall consider the primitive idempotent E_τ of the center Z_0 of the group ring of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$ which is associated with B_τ: $E_\tau = \sum_{\chi_i \in B_\tau} e_i$. Interpreting each class K_ν as the sum of all its elements, we may set

$$E_\tau = \sum{}' b_\nu^\tau K_\nu \tag{5}$$

with $\mathfrak{p}$-integral coefficients b_ν^τ, where the sum extends only over p-regular classes. We may also set

$$K_\alpha E_\tau = \sum_{\beta=1}^{n} a_{\alpha\beta}^\tau K_\beta \tag{6}$$

where $a_{\alpha\beta}^\tau = \sum{}' b_\nu^\tau a_{\alpha\nu\beta}$ with $K_\alpha K_\nu = \sum_\beta a_{\alpha\nu\beta} K_\beta$, hence $a_{\alpha\beta}^\tau$ are all in $\mathfrak{o}_\mathfrak{p}$. If n_ν is the order of the normalizer $\mathfrak{N}(G_\nu)$ of a representative element G_ν of K_ν

in $\mathfrak{G}$, then

$$a_{\alpha\beta}^{\tau} = \frac{1}{n_\alpha} \sum_{\chi_i \in B_\tau} \chi_i(G_\alpha)\,\bar{\chi}_i(G_\beta). \tag{7}$$

The following lemma is easily seen from (7).

Lemma 1. *If the p-element P belongs to the center of $\mathfrak{G}$, then $a_{\alpha\beta}^{\tau}$ can be different from zero only when either both K_α and K_β are contained in $\mathfrak{S}(P)$ or when both are not contained in $\mathfrak{S}(P)$.*

Let $\tilde{E}_\sigma$ be the primitive idempotent of the center $\tilde{Z}_0$ of the group ring of $\tilde{\mathfrak{G}}$ over $\mathfrak{o}_\mathfrak{p}$ which is associated with a block $\tilde{B}_\sigma$ of $\tilde{\mathfrak{G}}$. For each block B_τ of $\mathfrak{G}$, we shall consider the idempotent $\tilde{E}^{(\tau)}$ of $\tilde{Z}_0$ which is the sum of idempotents $\tilde{E}_\sigma$ with $\tilde{B}_\sigma^{(\mathfrak{G})} = B_\tau$; it is understood that if B_τ cannot be determined by any block $\tilde{B}_\sigma$ of $\tilde{\mathfrak{G}}$, then $\tilde{E}^{(\tau)} = 0$. We may set

$$\tilde{K}_\alpha \tilde{E}^{(\tau)} = \sum_{\beta=1}^{r} \tilde{a}_{\alpha\beta}^{(\tau)} \tilde{K}_\beta \qquad (\alpha = 1, 2, \ldots, r) \tag{8}$$

where $\tilde{a}_{\alpha\beta}^{(\tau)}$ are all in $\mathfrak{o}_\mathfrak{p}$. We then have

Lemma 2. *For $\alpha = 1, 2, \ldots, r$, we have*

$$a_{\alpha\beta}^{\tau} \equiv \tilde{a}_{\alpha\beta}^{(\tau)} \pmod{\mathfrak{p}} \qquad (\beta = 1, 2, \ldots, r). \tag{9}$$

Proof. Set $K_\beta^0 = \sum_{\tilde{K}_\varrho \subseteq K_\beta} \tilde{K}_\varrho$ and set $E_\tau^0 = \sum_\beta b_\beta^\tau K_\beta^0$. By a well known homomorphism (see (3) in [2] or (7B) in [4]), it is easily shown that (6) implies

$$K_\alpha^0 E_\tau^0 \equiv \sum_{\beta=1}^{n} a_{\alpha\beta}^{\tau} K_\beta^0 \pmod{\mathfrak{p} Z_0}. \tag{10}$$

Since $E_\tau^0 \equiv \tilde{E}^{(\tau)} \pmod{\mathfrak{p}\tilde{Z}_0}$ ([8], [5]), from (8) and Lemma 1 it follows

$$K_\alpha^0 E_\tau^0 \equiv \sum_{\beta=1}^{r} \tilde{a}_{\alpha\beta}^{(\tau)} P\tilde{K}_\beta + \sum_{\tilde{K}_\varrho \notin \tilde{\mathfrak{S}}(P)} *_\varrho \tilde{K}_\varrho \pmod{\mathfrak{p}\tilde{Z}_0}, \tag{11}$$

where the coefficients $*_\varrho$ are all in $\mathfrak{o}_\mathfrak{p}$. As for the right side of (10), we have

$$\sum_{\beta=1}^{n} a_{\alpha\beta}^{\tau} K_\beta^0 = \sum_{\beta=1}^{r} a_{\alpha\beta}^{\tau} P\tilde{K}_\beta + \sum_{\tilde{K}_\varrho \notin \tilde{\mathfrak{S}}(P)} *'_\varrho \tilde{K}_\varrho, \tag{12}$$

where the coefficients $*'_\varrho$ are all in $\mathfrak{o}_\mathfrak{p}$. By (10), (11) and (12), we obtain (9).

Let $\boldsymbol{\Phi}_P$ be the matrix $(\varphi_\varkappa^P(R_\alpha))$ of irreducible modular characters $\varphi_\varkappa^P$ of $\tilde{\mathfrak{G}}$, where each R_α is a representative element of $\tilde{K}_\alpha$ and where $\varkappa$ is the row index and α is the column index. Let further D^P be the matrix $(d_{i\varkappa}^P)$ of generalized decomposition numbers of $\mathfrak{G}$ belonging to P, where i is the row index and $\varkappa$ is the column index. If the characters χ_i and the modular characters $\varphi_\varkappa^P$

are arranged in a suitable order, then

$$D^P = \underbrace{\begin{pmatrix} D_0 & D_2 \\ D_1 & D_3 \end{pmatrix}}_{\text{associated with } \tilde{B}^{(\tau)}} \Big\} \text{ associated with } B_\tau$$

$$C_P = \underbrace{\begin{pmatrix} C_0 & 0 \\ 0 & C_1 \end{pmatrix}}_{\text{associated with } \tilde{B}^{(\tau)}} \Big\} \text{ associated with } \tilde{B}^{(\tau)}$$

By (7) for $G_\alpha = PR_\alpha$ and $G_\beta = PR_\beta$, we have

$$a^\tau_{\alpha\beta} = \frac{1}{\tilde{n}_\alpha} \sum_{\varkappa,\lambda=1}^{r} \sum_{\chi_i \in B_\tau} \varphi^P_\varkappa(R_\alpha)\, d^P_{i\varkappa}\, \bar{d}^P_{i\lambda}\, \bar{\varphi}^P_\lambda(R_\beta), \tag{13}$$

where $\tilde{n}_\alpha$ is the order of the normalizer $\mathfrak{N}(PR_\alpha)$ of PR_α in $\mathfrak{G}$ or that of the normalizer $\tilde{\mathfrak{N}}(R_\alpha)$ of R_α in $\tilde{\mathfrak{G}}$. Therefore if we set

$$A_\tau = (a^\tau_{\alpha\beta}) \qquad (\alpha, \beta = 1, 2, \ldots, r)$$

then, by the orthogonality relations for modular characters, we have

$$A_\tau = \bar{\Phi}_P^{-1} \begin{pmatrix} C_0^{-1} D_0' \bar{D}_0 & C_0^{-1} D_0' \bar{D}_2 \\ C_1^{-1} D_2' \bar{D}_0 & C_1^{-1} D_2' \bar{D}_2 \end{pmatrix} \bar{\Phi}_P. \tag{14}$$

Similarly if we set

$$\tilde{A}^{(\tau)} = (\tilde{a}^{(\tau)}_{\alpha\beta}) \quad (\alpha, \beta = 1, 2, \ldots, r)$$

then

$$\tilde{A}^{(\tau)} = \bar{\Phi}_P^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \bar{\Phi}_P. \tag{15}$$

Since $\det \bar{\Phi}_P \not\equiv 0 \pmod{\mathfrak{p}}$ and all $a^\tau_{\alpha\beta}$ are $\mathfrak{p}$-integers, from (14) it follows that the matrix $C_0^{-1} D_0' \bar{D}_0$ has $\mathfrak{p}$-integral coefficients. Therefore, comparing (14) with (15), by Lemma 2 we obtain (4).

Having obtained (4), we can finish the proof of Theorem 1 according to the arguments carried out in [5], §6.2. For the sake of completeness, we shall continue the proof after R. BRAUER [5].

Since all the p-conjugates of a character χ_i appear in the same block B_τ of $\mathfrak{G}$ (Theorem 6 in [1]), the positive definite Hermitian matrix $D_0' \bar{D}_0$ has integral rational coefficients. Further, as is easily seen, all coefficients of $C_0^{-1} D_0' \bar{D}_0$ are rational integers, hence $\det D_0' \bar{D}_0 \geqq \det C_0$. On the other hand, if we assume $D_1 \neq 0$ then $\det D_0' \bar{D}_0 < \det C_0$, because $D_0' \bar{D}_0 + D_1' \bar{D}_1 = C_0$ (Theorem 4 in [1]). Hence we must have $D_1 = 0$. Thus the proof is finished.

The following Corollary will be used in the next section.

COROLLARY. (BRAUER) *If L and M are two elements of $\mathfrak{G}$ which belong to different sections, then*

$$\sum_{\chi_i \in B_\tau} \chi_i(L)\, \bar{\chi}_i(M) = 0 \tag{16}$$

for each block B_τ of $\mathfrak{G}$ ([3], [5]).

2. Consequences. By Theorem 1, (14) becomes

$$A_\tau = \bar{\Phi}_P^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \bar{\Phi}_P, \tag{17}$$

and the combination of (17) with (15) yields

$$A_\tau = \tilde{A}^{(\tau)}. \tag{18}$$

From (7) and Corollary to Theorem 1, it follows

$$a_{\alpha\beta}^{\tau} = 0 \quad (\alpha = 1, 2, \ldots, r;\ \beta = r+1,\ r+2, \ldots, n). \tag{19}$$

By (18) and (19), Theorem 1 and its Corollary are formulated in terms of idempotents as follows:

THEOREM 2. *For $\alpha = 1, 2, \ldots, r$, we have*

$$K_\alpha E_\tau = \sum_{\beta=1}^{r} a_{\alpha\beta}^{\tau} K_\beta \tag{20}$$

and

$$\tilde{K}_\alpha \tilde{E}^{(\tau)} = \sum_{\beta=1}^{r} a_{\alpha\beta}^{\tau} \tilde{K}_\beta \tag{21}$$

with common coefficients $a_{\alpha\beta}^{\tau}$.

The following Corollaries 1 and 2 are seen immediately from (20).

COROLLARY 1. *If $\sum_{\nu=1}^{n} a_\nu K_\nu E_\tau = 0$, $a_\nu \in \Omega$, then $\sum_{K_\alpha \leqq \mathfrak{S}(P)} a_\alpha K_\alpha E_\tau = 0$ for each section $\mathfrak{S}(P)$ in $\mathfrak{G}$.*

COROLLARY 2. *If $\sum_{\nu=1}^{n} a_\nu K_\nu E_\tau \equiv 0 \pmod{\mathfrak{p} Z_0}$, $a_\nu \in \mathfrak{o}_\mathfrak{p}$, then $\sum_{K_\alpha \leqq \mathfrak{S}(P)} a_\alpha K_\alpha E_\tau \equiv 0 \pmod{\mathfrak{p} Z_0}$ for each section $\mathfrak{S}(P)$ in $\mathfrak{G}$.*

Corollary 2 is equivalent to the general case of (5 C) in [5] and Corollary 1 is equivalent to each of the following Corollary 3, Lemma in [9] and the general case of (5 B) in [5].

COROLLARY 3. *If a linear relation $\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = 0$, $a_\nu \in \Omega$, holds for every $\chi_i \in B_\tau$, then $\sum_{K_\alpha \leqq \mathfrak{S}(P)} a_\alpha \omega_i(K_\alpha) = 0$ for these χ_i, where ω_i is the linear character of Z belonging to χ_i.*

From Corollary 3 we see the following orthogonality relations for characters.

COROLLARY 4. (BRAUER, OSIMA) *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different blocks of $\mathfrak{G}$ then*

$$\sum_{G \in \mathfrak{S}(P)} \chi_i(G)\, \bar{\chi}_j(G) = 0 \tag{22}$$

for each section $\mathfrak{S}(P)$ in $\mathfrak{G}$ ([9]).

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Neuere Methoden und Ergebnisse der Ergodentheorie

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Dieser Band berichtet über die seit dem Erscheinen des Berichtes von E. HOPF (1937) erreichten Forschungsergebnisse. Die zentralen Resultate sind eingehend dargestellt und dienen als Ausgangspunkt für Literaturberichte in lockerer Form; hier wurde eine gewisse Vollständigkeit angestrebt. In zwei Anhangskapiteln findet sich der benötigte funktionalanalytisch-maßtheoretische Apparat; er soll vor allem denjenigen Lesern dienen, die in die Ergodentheorie neu eindringen wollen. Dem selben Zweck dient eine ausführliche Einleitung, in welcher die leitenden Ideen und Probleme behandelt werden, sowie einzelne Paragraphen in verschiedenen Kapiteln, die fast ohne spezielle Kenntnisse gelesen werden können.

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Mit Anwendungen auf technische Probleme

Von Dr. Ir. J. P. SCHOUTEN,
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Mit einfachen Beispielen anfangend wird die Operatorenrechnung im Geiste von OLIVER HEAVISIDE aufgebaut. Sie gibt dem Ingenieur ein wichtiges Werkzeug in die Hand, komplizierte Vorgänge in elektrischen Netzwerken oder mechanischen Systemen leicht zu übersehen und den Verlauf dieser Vorgänge beim Entwurf seiner Schaltungen oder Systeme vorauszuberechnen. Es wird dem Ingenieur gezeigt, wie die Regeln und Rechenvorschriften, die HEAVISIDE gegeben und angewendet hat, ohne Furcht vor Fehlern benutzt werden können, wenn man sie nur passend ergänzt.

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